

Estimation of liquidity-adjusted VaR from historical data

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PD Dr. Marlene Müller

Institute for Statistics and Econometrics

CASE-Center for Applied Statistics and Economics

Humboldt-Universität zu Berlin



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Ekaterina Orlova

(513459)

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1 Introduction

Risk is connected with deviation of actual outcome from expected one in the adverse direction for the agent. Nowadays VaR measure, that was initially developed for measuring market risk, is used also for control and regulation purposes, as well as in other areas. Market risk itself arises from the changes in level or volatility of market prices, and mid-prices are used for VaR evaluation. However, this approach causes questions, if it is assumed that portfolio of assets is liquidated, because the transaction will be hold not at mid-price. The real price will depend on the ability of transaction's volume to influence existing spread and on the value of the spread itself, so that liquidity of the market begins to play the role. The reason for turning to the consideration of liquidity risk is VaR underestimation under usual framework in this situation, and the underestimation will lead to the increase of the market risk capital requirements, as they are connected with multiplication factor, determined by number of VaR violations. Thus, if VaR is significantly underestimated, it will have consequences from the regulator side. The significance of underestimation will depend on the liquidity of liquidated portfolio.

There are number of studies that are devoted to incorporation of liquidity risk into VaR model. These studies are divided into two broad classes: one researchers develop the models of endogenous liquidity risk incorporation, when this type of risk is unique for the agent and presents the effect of liquidated quantity on the prices, other authors consider exogenous liquidity risk, which corresponds to the existing spread on the market. Moreover, some extensions were suggested in order to combine these two types of liquidity risk in one model.

In our work liquidity-adjusted VaR, accounting for exogenous liquidity risk, is estimated for highly liquid and less liquid portfolios. Two portfolios were used as they enable to show the difference in significance of liquidity component and to conclude about the relative importance of using liquidity-adjusted VaR instead of ordinary one.

As VaR can be estimated with different methods and many modifications of these approaches exist, allowing to overcome some drawbacks of initial one, part of the work is devoted to description of three main methods (variance-covariance approach, historical simulation method and Monte Carlo method) and their extensions, which we will also use in order to estimate the model of our interest.

After the model is estimated the natural question of interest is whether the chosen model is accurate. In order to response to this question backtesting procedure has to be applied to results of estimation. In the literature different tests were suggested, which enable to verify the accuracy of the model according to certain points. We used three tests here (unconditional coverage test, test of independence and joint test) and compared results of backtesting for ordinary VaR and liquidity-adjusted VaR, obtained on the basis of different methods. In addition, as VaR is not a coherent risk measure, CVaR was estimated for two portfolios.

The work is organized in the following way: the second section is referred to description of VaR concept, then (section 3) the review of possible methods of incorporation of liquidity risk into VaR model is presented, as well as more detailed consideration of characteristics of liquidity risk; one of this models will be used further in our empirical analysis. Section 4 is devoted to description of methods of VaR estimation, their strong and weak points, possible improvements, and possibility of application to liquidity-adjusted VaR estimation. In section 5 different tests for model's verification are described. Then we turn to empirical analysis, and all results of estimation and backtesting are presented in the section 6. In addition, the concept of CVaR is introduced in section 7 and results of estimation are presented. Finally, the conclusion concerning importance of liquidity-adjusted VaR and adequate methods is made.

2 VaR concept

Value at risk (VaR) presents the maximum losses that can occur over the given time horizon with certain probability. Thus, VaR is equaled to the value that will not be exceeded over given time horizon with some probability and answers the next question: what is the maximum loss for the given time horizon that with small probability (for example, 0.01) actual losses will be higher than this value. Consequently, VaR will be exceeded with some frequency.

There are two types of VaR that can be estimated: relative VaR, when the loss is defined relative to expected value, and absolute VaR, when the loss is compared with initial position. Jorion (2001) formulates this difference in the next way. If the sum of initial position is W_0 , then the value of position at the end of the period is $W = W_0(1 + R)$, where R is return with $E(R) = \mu, V(R) = \sigma^2$.

The worst possible return for certain confidence level is denoted as R^* , then relative and absolute VaR are defined, respectively, by following expressions:

$$\begin{aligned} VaR &= E(W) - W^* = -W_0(R^* - \mu) \\ VaR' &= W_0 - W^* = -W_0R^*, \end{aligned}$$

where $W^* = W_0(1 + R^*)$ present the worst possible portfolio value.

According to the definition we want to find the worst possible portfolio value that will not be exceeded with some probability: $P(w \leq W^*) = 1 - c$, where c is confidence level. It means that W^* is quantile of distribution of portfolio value. If normal distribution is assumed then next results are obtained for relative and absolute VaR.

Worst return can be found using standard normal distribution:

$$P(R < R^*) = P(Z < \frac{R^* - \mu}{\sigma}) = 1 - c,$$

where $Z = \frac{R - \mu}{\sigma} \sim N(0, 1)$, consequently $R^* = \mu + \alpha\sigma$, where $\alpha < 0$ is quantile of standard normal distribution. Thus, using formulas above, relative VaR is equaled to $VaR = -\alpha\sigma W_0$, absolute VaR is written as $VaR' = -(\alpha\sigma + \mu)W_0$.

Usually portfolio is valued on the basis of mid prices, regardless the fact that the volume of transaction itself can influence existing price or just the fact, that the real price of transaction accounts for spread on the market and depends on whether the asset is bought or sold. The ignorance of the latter can lead to underestimation of risk resulting in the lower VaR value. This, in its own turn, will lead to higher number of VaR violations by real losses, meaning that market risk capital requirements will be increased. The connection between number of VaR violations and market risk capital requirements will be considered in details further, now only the idea of importance of deviation of real price from mid price is presented. The next section is devoted to description of two types of liquidity risk and overview of methods of incorporating these types of liquidity risk into VaR model, one of this methods will be used then for our empirical calculations.

3 Literature review

3.1 Types of liquidity risk

Liquidity risk is one of types of financial risk and can be of great importance to financial institutions, as the history of LTCM has shown. In most general way the liquidity market can be defined as market, where market participants can quickly conduct transactions of big volume without significant influence on price. Liquidity risk itself can be divided into two groups: **market liquidity risk** and **funding liquidity risk**. The former appears when the real price of transaction differs from the market price, the latter assumes that company cannot meet its financial obligations (the ability to meet obligations strongly depends on the structure of assets and liabilities of the company, because in case of having short-term liabilities the company will have difficulties with their implementation if there are no high-liquid assets that can be easily transferred into cash). But we will focus here on the market liquidity risk.

Mid prices present the average values between bid and ask prices, and are used for VaR calculation. However, this approach is not appropriate in reality, as the price of transaction differs from the mid price - the sale is implemented with respect to bid price, the purchase-with respect to ask price. Moreover, if the volume of position exceeds the normal market size, then bid and ask prices move in adverse direction for the trader, so that if the trader is liquidating large position, then bid price will be falling in some way after the traded quantity exceeds the normal market size. Thus, the market liquidity risk can be divided into **exogenous liquidity risk**, associated with observed bid-ask spread, and **endogenous liquidity risk**, connected with influence of liquidated quantity on the price of the asset. One way to deal with market liquidity risk is to set limits on positions in portfolio, as it can enable to escape sufficient losses when the necessity of portfolio liquidation appears. Above-described idea of exogenous and endogenous liquidity risks is presented on the graph (see Figure 1), where the spread is showed and the movement of bid and ask prices in adverse direction after some point.

The market can be characterized as deep market or thin market according to the level of impact of sales on price (if the influence of traded quantity on price is not significant and the realized spread does not differ much from the observed one, then the market can be referred to category of deep markets; if the effect on price is large enough, then the market is thin). As example of deep markets the markets of high-liquid securities (such as Treasury bonds, main currencies) can be considered; the depth itself reflects the activity of participants of the market, volume of trading. Another two characteristics of liquidity of the market are tightness and resiliency. Tightness shows how far the price of transaction deviates from the mid price, resiliency reflects the time, necessary for the price to recover after the transaction was conducted.

As in certain models that will be considered below, spread is used in order to account for liquidity component in VaR, it will be useful to look at the concept of spread in

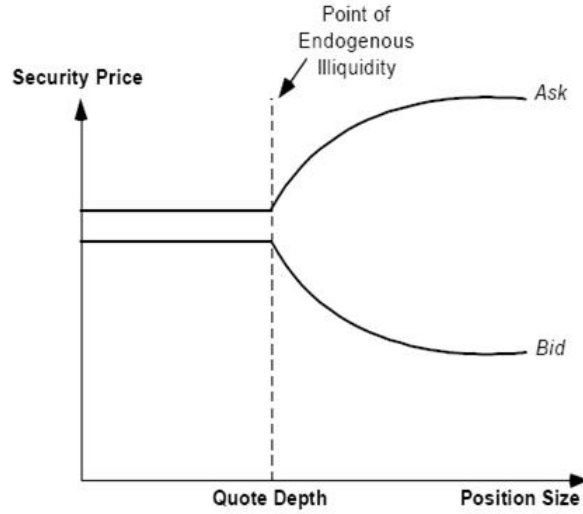


Figure 1: Effect of position size on liquidation value. *Source: Bangia et al. (1999)*

more details. Jorion (2001) points out that spread reflects three types of costs: order processing costs (these costs are associated, for example, with state of technology, cost of trading), asymmetric information costs (they are referred to orders coming from informed traders) and inventory-carrying costs (present the costs of maintaining open positions). Models, associated with spread, can be used for incorporating exogenous and endogenous liquidity risk in VaR framework. Now we turn to the review of studies, which were conducted in order to find the methods of including liquidity risk in VaR model.

3.2 Liquidity risk and VaR

These researches can be divided into two broad classes. First, there are models, which consider the problem of accounting for the endogenous liquidity risk by searching for optimal liquidation strategies of position. It is important as immediate liquidation of position results in high costs, but in case of slow liquidation the position exposed to price risk, so there is a trade-off between execution costs and price risk and the problem of finding optimal trading strategy appears. The latter can be done by minimizing transaction cost or maximizing expected revenue from trading, then, basing on received optimal strategy, liquidity-adjusted value at risk can be derived. Second class of models is devoted to modeling exogenous liquidity risk through studying the distribution of spread. In addition, certain modifications allow to include endogenous liquidity risk in this class of models. But before we will start with models, presenting approaches of the first group, it should be mentioned about *ad hoc* way of adjusting

VaR to liquidity risk.

One of the most simple ways of introducing liquidity risk in VaR model is to adjust the time horizon of VaR according to inherent liquidity of portfolio. This ad hoc approach does not enable to reach the goal it is aimed at. In spite of adjusting the time horizon to the inherent liquidity of portfolio, the calculation of value at risk assumes that the liquidation of all position is taken at the end of the holding period, and is not taken orderly during the period.

Shamroukh (2000) suggests the model, where the liquidation of portfolio is taken orderly throughout the holding period, thus, the liquidation-adjusted value at risk is obtained. Author begins with the model for one asset and one risk factor. The main idea is to calculate the mean and variance of portfolio value defined when the liquidation is over, but important point here is that portfolio is liquidated by parts during the holding period. The initial position is assumed to be uniformly liquidated over the period T (at time T the liquidation is completed). The liquidation schedule is characterized by the sequence of trade dates and volumes of trading. The logarithm of ratio of risk factor's levels is assumed to be normally distributed, portfolio value at time T can be computed as the sum of products of sold number of units of asset and the price of sale. After certain transformations the variance of portfolio value is received and on its basis liquidation-adjusted value at risk can be found (it is computed as usual value at risk, but due to the fact that liquidation is taken throughout the holding period, the variance differs from the ordinary case, thus, obtained value at risk also differs from standard RiskMetrics VaR). The difference between two measures represents liquidation factor, it depends on the number of trading dates. If number of trading dates tends to infinity, then liquidation factor tends to $1/3$. Author also extends this model to the case of portfolio of multiple assets which are influenced by multiple risk factors. More complex derivations lead to same result in relation between liquidation-adjusted value at risk and the usual one. Then author introduced exogenous and endogenous liquidity costs by constructing the liquidation price of the asset (endogenous liquidity cost presents the sensitivity of liquidation price to trade size). This liquidation price is used in calculation of portfolio value at time T , thus, liquidation-adjusted and liquidity-cost adjusted value at risk (LA-VaR) is obtained. The holding period can be then considered as endogenous variable and found as output of the model. The liquidation schedule defines the level of VaR and author offers to consider the minimal of these values as LA-VaR: for some given trading frequency the number of trading dates that minimizes derived VaR can be found. Then, by definition, liquidation period T is computed as product of trading frequency and optimal number of trading dates.

3.2.1 Incorporation of endogenous liquidity risk into VaR model

One of the basic studies, devoted to finding optimal liquidation strategy, and defining liquidity-adjusted VaR on its basis, is the study of Almgren and Chriss (1999), who introduce the notion of liquidity-adjusted VaR in the framework of choosing the

optimal strategy of portfolio liquidation. Authors consider trading model, where the initial portfolio consists of block of X units of security (extension for portfolios exist in this model, but we will turn to it later) and has to be liquidated by the fixed time T in the future (further we will talk in terms of shares, but also futures contracts and units of currency are considered as securities in the model). The whole time interval is divided into N small intervals of length τ , in which the liquidation of shares takes place, so that at time T number of holding shares in portfolio is zero. The trading trajectory $x = (x_0, x_1, \dots, x_N)$ represents number of shares that will be hold at discrete times $t_k = k\tau, k = 0, \dots, N$. In addition, trade list is also defined, it represents number of shares (n_1, \dots, n_N) that are sold during small intervals and, consequently, each number equals to the difference between adjacent points of trading trajectory. Another variable, which is constructed, is average rate of trading; it is defined as the ratio of quantity traded in the time interval to the length of time interval itself: $v_k = \frac{n_k}{\tau}$. The price of the stock is assumed to follow discrete arithmetic random walk:

$$S_k = S_{k-1} + \sigma\tau^{1/2}\xi_k - \tau g\left(\frac{n_k}{\tau}\right), k = 1, \dots, N, \quad (1)$$

where σ is the stock's volatility, ξ_i -independent random variables (with zero mean and unit standard deviation), $g(v)$ is a function of the average rate of trading. This function is permanent market impact function.

Authors consider the influence of sale of shares on the stock's price through functions of permanent and temporary market impact. Permanent market impact is the impact of trades on the market price, the main feature of which is that, once occurred, it lasts until the portfolio is liquidated. The function of permanent market impact can be linear in the average rate of trading: $g(v) = \gamma v$, so that it depicts the decrease in the stock's price per unit time due to selling of shares at the average rate of trading. Thus, in order to include the resulting effect of selling certain number of shares in one time interval on the stock's price, the sold number of shares has to be multiplied by the coefficient of proportionality γ .

In contrast to permanent market impact, temporary market impact exists only in the period, when liquidation of the certain block of shares takes place: selling of n_k shares in the interval between t_{k-1} and t_k influences the price only in this time interval and does not influence the price in consequent time intervals. Hisata and Yamai (2000) note, that in order temporary market impact disappears in the next period, the price of stock has to increase by the value of temporary market impact in order only permanent market impact remains to the beginning of next period. Temporary market impact function also can be assumed to be linear function of average rate of trading, having additional term that represents fixed costs of selling (as an example of fixed costs authors cite half of bid-ask spread and fees): $h(v) = \epsilon \cdot \text{sgn}(n_k) + \eta v$, where ϵ -fixed costs of selling, sgn -sign function. This expression corresponds to decline in price per share; if n shares are sold, then full effect of temporary market impact will equal (in correspondence with definition of average rate of trading) $nh(\frac{n}{\tau}) = \epsilon|n| + \frac{\eta}{\tau}n^2$, so that total costs are quadratic in number of

shares sold. Accounting for temporary impact of trades on the price, the latter can be written in the next way (in general case):

$$\tilde{S}_k = S_{k-1} - h(v_k) \quad (2)$$

Using equations (1) and (2) authors deduce the trading revenue (so called capture of the trading trajectory) that presents the sum of products of number of sold shares and the price of sale, expression for which was obtained before:

$$\sum_{k=1}^N n_k \tilde{S}_k = XS_0 + \sum_{k=1}^N (\sigma\tau^{1/2}\xi_k - \tau g(\frac{n_k}{\tau}))x_k - \sum_{k=1}^N n_k h(\frac{n_k}{\tau})$$

Thus, the difference between the initial value of portfolio (XS_0) and its liquidation value ($\sum n_k \tilde{S}_k$) can be found, this difference represents the total cost of trading (it is also considered as measure of transaction costs) and is called implementation shortfall. According to the assumptions of the model, it is random variable (if $\xi_i \sim N(0, 1)$, then implementation shortfall is also normally distributed). Mathematical expectation and variance of total cost of trading can be calculated, these two moments depend on the trading trajectory x and are marked as $E(x), V(x)$ respectively. For example, if all shares are sold in the first time interval, then variance is zero and mathematical expectation of total cost of trading increases with increase in number of time intervals.

As mathematical expectation and variance depend on the chosen trading trajectory, then the question of optimal trading trajectory appears. For the given value of variance trader will chose the trading strategy that minimizes expected cost (this problem of constrained minimization is solved with help of Lagrange multiplier λ , that reflects the risk-aversion of the agent). Consequently, in the coordinate $(V(x), E(x))$ the efficient frontier of optimal trading strategies can be build. In order to choose the trading strategy from those, composing the efficient frontier, one can use the utility function approach or look at value at risk.

In the first case, the coefficient of risk aversion, which is determined by utility function, is used instead of Lagrange multiplier, while the minimization problem remains the same. In the second case, authors apply the concept of value at risk to the total cost of trading, so that value at risk is defined as level of transaction costs that will not be exceeded with probability p :

$$VaR_p(x) = \lambda_v \sqrt{V(x)} + E(x),$$

where λ_v is the quantile of standard normal distribution corresponding to the certain level of significance. As we can see, value at risk depends on the trading strategy x . Trading strategy x is called efficient if it allows to get the minimum possible value at risk for the given level of significance $(1 - p)$. Authors call this minimum possible value at risk L-VaR. It means that liquidity-adjusted value at risk is defined as value at risk for the optimal strategy x , and optimality of the latter is related to

minimization of value at risk for given level of significance and given holding period T .

Authors also extend the model for portfolio of assets. The idea is the same as in case of one asset, but now stock prices follow multidimensional arithmetic Brownian random walk, instead of coefficients of proportionality in permanent and temporary market impact functions matrices begin to depict the influence of trading on prices. As in previous case, mathematical expectation and variance of total costs of liquidation can be computed, then the optimal trading strategy can be found.

Hisata and Yamai (2000) continue the research of Almgren and Chriss and consider the problem of finding optimal execution strategy, but in the case of endogenous holding period with the assumption of sales at constant speed. Authors use practically the same model of price movement: permanent and temporary market impact functions are included in the model of price movement (however, the sales price at time k is determined by deduction of temporary market impact function from the price of that period, whereas Almgren and Chriss deduct this function from the price of the previous period). On the basis of given model of price movement transaction costs are found as difference between initial value of the position and liquidation value. Then mathematical expectation and variance of transaction costs are derived. On their basis the function, which has to be minimized in order to obtain the optimal execution strategy, is built. It represents the sum of mathematical expectation of transaction costs and the product of multiplication of standard deviation of transaction costs, cost of capital r and certain percentile of standard normal distribution (the latter is determined by investor's risk aversion). While the first term of the sum presents the average change in the value of position, the second term reflects the influence of market risk. Minimization of the described function under condition of sales at constant speed with respect to number of sales enables to find the optimal number of sales, and consequently, the optimal holding period. Then liquidity-adjusted VaR can be defined: it is defined as relative VaR and equals the product of percentile of standard normal distribution for given confidence level and standard deviation of transaction costs which occur in case of optimal trading strategy. Authors suggest also different extensions of the model such as continuous time model, stochastic market impact model, extension for portfolio of assets in the continuous time framework.

Berkowitz (2000) suggests to account for liquidity risk in usual VaR framework by considering the influence of amount of sold assets on prices, and on the basis of these prices to estimate portfolio value. The value of portfolio is supposed to be determined by positions in assets and pricing function which defines the effect of risk factors on the portfolio value. But, as it is known, changes in asset price are connected with the changes in volume of the position in this asset, so that the downward demand curve for the asset is observed (author uses the concept of elasticity of demand). The negative slope can be explained on the basis of theory of asymmetric information: selling large amounts of asset can be considered as signal that informed agents try to deliver from it due to their private knowledge. Thus, the effect of selling asset on its price is included in the process of the price movement in the next way: the

influence is linear and the total effect presents the negative value of amount of asset sold multiplied by some parameter (in the capacity of assets shares are considered below, the estimation of the parameter will be described later). The manager of portfolio of shares faces the problem of maximizing expected revenue from trading over the whole holding period subject to condition that sum of traded shares has to be equal to given number of shares. The price of the following period equals the price of the previous period adjusted to the market-wide change in price of the share and the above-described term presenting the influence of amount of sold shares on the price. The optimal number of trading shares is found from the maximization problem (the solution for optimal number of trading shares obtained by Bertsimas and Lo ¹ is used). Then, the solution is plugged into the equation which defines the process of price movement, and consequently, the portfolio value can be obtained. The latter appears to consist of two terms: one term is responsible for market risk component and corresponds to the price of the previous period and the market-wide change in price, another term reflects the reaction of price on the amount of asset sold, the effect of influence of liquidating position on the price. The mathematical expectation and variance of the portfolio value can be found (the market-wide change in price and number of shares sold are assumed to be independent, this leads to additional term in the expression for variance). The parameter in the equation for price movement is obtained as the estimation from regression, where dependent variable is difference in prices between two periods. Thus, the calculation of value at risk is based on the rebuilding of portfolio values which account for decrease of price from optimal investor's sales. Author also points out that the distribution of portfolio values can be estimated by numerical methods.

Jarrow and Subramanian (2001) paid attention not only to market impact of sales on the price of asset, but also to the existence of execution lag, so that the sale is not executed immediately after the order arrives. These two points are considered as features of liquidity, the case of absence of execution lag and market impact is the case of absence of liquidity risk. In the model the price of stock follows geometric Brownian motion, the impact of sales on price is included with help of price discount function which owns certain properties (one of properties is that the function is nonincreasing in sales) and existence of execution lag is defined by nondecreasing function of sales (the latter means that the larger the sale is, the more time it will take to execute this order). The aim of the trader, who has some number of shares, is to find such strategy of liquidation that will maximize the expected revenue from sale. Authors found out, that if the trader is price taker (the case of no liquidity risk), then the optimal trading strategy for him is block liquidation of assets. Depending on whether the drift in the price process is positive or negative, the block liquidation has to be taken, respectively, at terminal date or immediately. In case of liquidity risk the optimal execution strategy will be the same as in previous case only if the condition

¹Bertsimas and Lo (1998) solve the problem of minimization of expected cost of selling large block of equity over a fixed time horizon by deriving dynamic optimal trading strategies. The optimal execution strategy turns to be function of market conditions.

of economies of scale in trading holds. This condition provides that the cumulative price discount in case of selling all shares in two parts is less or equal than the price discount in case of selling all shares in one time. The liquidity discount is computed then as difference between market price of the share and its liquidation value. The calculation of liquidity-adjusted value at risk, based on this model, requires knowledge of average and standard deviation of price discount for the number of shares sold and of the execution period, but there are no available data that can be used for estimation of necessary parameters.

All described models were dealing with endogenous liquidity risk, however, it is not so easy to apply these methods in practice due to lack of necessary data and difficulties in determining some parameters of the models (for example, coefficient of proportionality of temporary market impact function). On the contrary, the model, which is described below, can be evaluated on the basis of available data.

3.2.2 Incorporation of exogenous liquidity risk into VaR model

Bangia, Diebold, Schuermann and Stroughair (1999) proposed the model for incorporation of exogenous liquidity risk into VaR model. Authors make strong distinction between exogenous liquidity risk, which is similar to all market participants, cannot be influenced by the actions of one player and presents the market characteristics, and endogenous liquidity risk, which is special for each player according to the volume of trading position, as after the volume exceeds the level of quote depth, the influence of traded size on bid and ask prices occurs. The main idea of including the exogenous liquidity risk in the VaR model is that in case of not perfectly liquid markets the liquidation of position is executed not at mid price, but this price has to be adjusted for the value of existed spread. Thus, as in order to compute usual VaR the worst price of asset for some confidence level is considered, then in order to account for effect of spread on the price of transaction in the VaR calculation, the worst value of spread for certain confidence level has to be considered. Below the model itself is described.

One-day asset return is defined as the logarithm of the ratio of two adjacent prices and assumed to be normally distributed with mathematical expectation $E(r_t)$ and variance σ_t^2 :

$$r_t = \ln \left(\frac{P_t}{P_{t-1}} \right) \sim N(E(r_t), \sigma_t^2)$$

For given confidence level (authors use confidence level of 99 %, the corresponding quantile equals 2.33) the worst return can be found and, consequently, the worst price of the asset:

$$P_w = P_t e^{E(r_t) - 2.33\sigma_t}$$

Authors consider one-day horizon; the expected daily return is taken to be equal to zero, then the parametric VaR can be written in the following way:

$$P - VaR = P_t(1 - e^{-2.33\sigma_t})$$

In the empirical analysis authors computed variance using exponential weighted moving average, as clustering effects are observed for time series of asset returns, when periods of large and small returns volatility are clustered and distinct from each other. It means that variance changes over time, and exponentially weighted moving average enables to capture for this change.

As the next step, authors turn to spread behavior in order to include its effect in the VaR framework. As in previous case we were interested in the worst price (for given confidence level), so now we are interested in the worst movement of spread. The exogenous cost of liquidity (COL) is determined in the following way:

$$COL = \frac{1}{2}[P_t(\bar{S} + a\tilde{\sigma})],$$

where \bar{S} is average relative spread ($S = \frac{Ask-Bid}{Mid}$), P_t is mid price of the asset, $\tilde{\sigma}$ is volatility of relative spread, a is scaling factor that has to provide the confidence level of 99 %. With latter parameter certain problems are connected: this parameter has to be evaluated empirically, because the distribution of spread is far from normal and there are no tables from which the values of parameter can be taken. The estimated interval for values of a is $[2; 4.5]$, exact number depends on the instrument and market.

The procedure of a estimation is based on the idea that the worst possible relative spread for some given confidence level can be computed using historical simulation method and using deviation from the mean relative spread: $\bar{S} + a\tilde{\sigma}$. The series of worst possible relative spreads estimated from historical simulation method is known, on the contrary, until the a factor is unknown, the worst possible relative spreads cannot be estimated from the second method. As one measure-worst possible relative spread-is obtained on the basis of two different methods, and the parameter, which is used in one of the methods, is not known, it can be estimated from the regression equation of known worst possible relative spreads from the first method (historical simulation method) on the worst possible relative spreads from the second method. Then, estimated a factor can be used for obtaining the exogenous cost of liquidity.

After the exogenous cost of liquidity, presenting the measure of exogenous liquidity risk, was derived, the assumption concerning the movement of prices and spreads is made: in adverse market environment extreme events in spreads and prices happen simultaneously. It means that if price has changed to its worst level for some given confidence level, then spread changed for its worst value too. This enables to write down the worst price of transaction in the next way:

$$P' = P_t e^{-2,33\sigma_t} - \frac{1}{2}[P_t(\bar{S} + a\tilde{\sigma})]$$

On the basis of previous expression, liquidity-adjusted VaR can be found:

$$LAdj - VaR = P_t(1 - e^{-2,33\sigma_t}) + \frac{1}{2}[P_t(\bar{S} + a\tilde{\sigma})]$$

Empirical studies show that distribution of returns is not normal and has fat tails. In order to deal with this fact, authors introduce parameter which will control for fat tails of returns distribution:

$$P - VaR = P_t(1 - e^{-2,33\theta\sigma_t})$$

If the distribution is normal, $\theta = 1$; θ increases with the increase in deviation of distribution from normality.²

All these derivations were done for single asset. However, it is possible to extend the model to *portfolio level*. Authors suggest to compute the second term in formula for $LAdj - VaR$ by finding spread for portfolio. The latter can be calculated on the basis of portfolio bid and ask series, which can be obtained as the weighted sum of series of bids and asks of portfolio's assets. Thus, in case of portfolio $LAdj - VaR$ is also calculated as sum of two terms: usual VaR and component reflecting the exogenous liquidity risk. It should be mentioned here, that another possible way of extending the model to portfolio level, is to redefine prices in correspondence with existing spread, and then use these new prices for VaR calculation. The model of Francois-Heude and Van Wynendaele (2001), which will be described here later, can be viewed as certain point of redefining the prices (the way of adjusting the mid price to the spread, proposed in their model, can be useful for the problem of extending current approach to portfolio level).

In the paper authors present also empirical results of model's estimation: they estimate the model for one asset case (data for currency exchange rates were used) using EWMA scheme for volatility calculation, estimation was conducted also for different portfolios. Liquidity component is more significant for less liquid markets and matters in determining the number of VaR violations and, consequently, the multiplication factor.

3.2.3 Exogenous and endogenous liquidity risk in VaR model

Le Saout (2002) applies the model of Bangia et al. to French stock market and extends the model in order to account for endogenous risk. Author substitutes the bid-ask spread which is used for value at risk calculation by Weighted Average Spread (WAS). WAS is connected with the market, where sale and purchase of large blocks of assets are allowed to be performed in one transaction and its price has to be in the interval, defined by WAS for block of standard size. The WAS presents the difference between weighted bids and asks: bids and asks are weighted according to the quantities pointed in the buy- and sell-orders (orders are added up in order to reach standard size of the block) and these weighted sums are divided then by the quantity, corresponding to

²Empirical relationship between parameter θ and kurtosis κ in case of t-distribution was derived: $\theta = 1.0 + \phi \cdot \ln(\kappa/3)$, where ϕ is constant, which can be estimated from regression equation: $P - VaR = P_t(1 - e^{-2,33\theta\sigma_t})$, using historical VaR, the value of parameter depends on the tail probability. After ϕ is obtained, it can be used for the estimation of correction factor θ .

the block's standard size. Thus, transaction with number of shares in the block equal or more than standard size will be taken at some price from the described interval. It means that now the second term in the formula for $LAdj - VaR$ incorporates also the influence of traded size on price of stock, accounts for endogenous risk. Empirical estimation of the part of $LAdj - VaR$, related to liquidity risk, changed in case of incorporation of endogenous liquidity risk in comparison with the case when only exogenous risk was included in $LAdj - VaR$. Component, responsible for liquidity risk, has increased after calculations were held with WAS.

The idea of using WAS as the mean of including endogenous liquidity risk in VaR framework is met also in the work of Francois-Heude and Van Wynendaele (2001). Authors criticize the model of Bangia et al. and suggest certain modifications, which allow to escape the main disadvantages of this model, among one of them is the problem of endogenous liquidity risk.

Authors emphasize four main disadvantages of the model of our interest: the necessity to estimate α parameter, as spread distribution is not normal; assumption that in adverse market environment extreme changes in prices and spreads happen simultaneously; lack of component of endogenous liquidity risk in the model and ignorance of dynamic aspect of liquidity. In order to overcome first two problems, the new way of incorporating exogenous liquidity risk in value at risk is suggested:

$$L - VaR_t = Mid_{BL_t} - \left(Mid_{BL_t} \cdot \left(1 - \frac{\bar{S}p_{BL}}{2} \right) \cdot e^{-\alpha\sigma} \right),$$

where Mid_{BL_t} is mid price at the best limit at time t , $\bar{S}p_{BL}$ is average relative spread. Thus, this way of introducing exogenous liquidity risk does not require consideration of distribution of spread and does not assume extreme changes in prices and spreads to happen simultaneously. In the proposed framework the mid price is adjusted to the existence of spread, so that the redefined price is used for searching the worst price (for some confidence level) and VaR. In order to account for dynamic aspect of liquidity, authors introduce the new term in the expression for $L - VaR_t$, which controls for difference between relative quoted spread and average relative spread:

$$L - VaR_t = Mid_{BL_t} - \left(Mid_{BL_t} \cdot \left(1 - \frac{\bar{S}p_{BL}}{2} \right) \cdot e^{-\alpha\sigma} \right) + Mid_{BL_t} \cdot \left(\frac{Mid_{BL_t} - B_{BL_t}}{Mid_{BL_t}} - \frac{\bar{S}p_{BL}}{2} \right)$$

The sign of this difference (the third term of expression) will increase or decrease $L - VaR_t$, and the difference itself can be viewed as volatility of liquidity level. And the last modification concerns inclusion of endogenous liquidity risk in the model: relative quoted spread and average relative spread have to be adjusted to the traded quantity³. Proposed model was applied to the intraday data (holding period was taken to be 15 minutes).

But in our work we will focus on the incorporation of exogenous liquidity risk in VaR model, and more precisely, on the model of Bangia et al. and its results, depending

³Authors argue about interpolation of bids and asks between quoted Average Weighted Spread and bid and ask prices at best limit.

on the method of calculation of liquidity-adjusted value at risk: historical simulation, variance-covariance approach and Monte Carlo method.

4 Methods of estimation

There are three main groups of methods that are usually used for VaR estimation: historical simulation, variance-covariance approach and Monte Carlo method. However, in each group there are number of modifications of basic method which enable to overcome some drawbacks and to account for special features of real data. Below the basic ideas of these three methods are described and also certain modifications of these approaches are considered.

4.1 Variance-Covariance method

This approach assumes the normal distribution of log-returns of risk factors and different methods of volatility calculation, which will be briefly described below.

Idea of delta-normal method is to approximate the change in portfolio value by changes in risk factors according to sensitivities of portfolio's value to these changes (in fact, the approach for single asset was described earlier, when the model of Bangia et al. was presented). Sensitivities are obtained from the first-order Taylor expansion of portfolio value ⁴ (that is why the method is also called the local valuation method: the portfolio is valued once, changes in its value are introduced by derivatives). As returns of risk factors are assumed to be normally distributed, portfolio return is also normally distributed. On the basis of derived expression for portfolio return, its variance is computed (using the covariance matrix of returns of risk factors). The variance is used then for VaR calculation: $VaR = \alpha \sqrt{x' \Sigma x}$, where α - quantile of standard normal distribution corresponding to given confidence level, x -vector of sensitivities of absolute change in portfolio value to returns of risk factors, Σ -covariance matrix of risk factors returns.

Advantages of this method are simplicity of implementation and computational speed. One of the most important shortcomings of the model is assumption of normal distribution of returns, as it is known that distribution of returns of financial assets has fat tails. Goorbergh and Vlaar (1999) investigate different methods of introducing fat tails in the model and test the accuracy of the models with Kupiec test (this test will be discussed later in the section backtesting). Authors study the effect of assuming t-distribution of log-returns of portfolio, results of VaR calculation using the mixture of normal distributions with different variances. They also consider a class of models with time-varying volatility in order to capture clustering effects.

Duffie and Pan (1997) also studied the problem of fat tails and estimation of current volatility that will be used for VaR calculation. Authors emphasize two sources of fat tails: jumps and stochastic volatility. They consider next model of returns:

⁴If portfolio contains non-linear instruments, then derivatives of higher order have to be taken. The method is called then delta-gamma approximation, but as we will use stocks for further calculations, we are not interested in this method here.

$r_{t+1} = \mu_t + \sigma_t \epsilon_{t+1}$, where μ_t - expectation of return r_{t+1} , conditional on the information available at day t ; σ_t - standard deviation of r_{t+1} , conditional on the information available at day t ; ϵ_{t+1} - shock, its conditional mean equals zero, conditional standard deviation equals one. Estimated VaR is higher when jumps are introduced in the distribution of shocks in comparison with the case when shocks are normally distributed. Authors present different models of stochastic volatility: regime-switching volatility, when volatility behaves according to a finite-state Markov chain; autoregressive volatility; GARCH and exponential GARCH (EGARCH) models. In fact, there are many studies, devoted to the problem of choosing the best model of volatility forecasting that can be used in VaR calculation.

For example, Polasek and Pojarliev (2000) studied the performance of different volatility models on the basis of Christoffersen test (this test also will be described in the section backtesting), assuming that returns of NASDAQ 100 Index are normally distributed (the data, that were used). Authors compared the accuracy of VaR estimates computed according to such volatility models as sampling variance, RiskMetrics model, GARCH model, t-GARCH, asymmetric GARCH model, EGARCH, power GARCH. Among all approaches GARCH model appeared to be the best one. But here we will focus on three main and commonly used methods of volatility calculation: equally weighted moving average, exponentially weighted moving average and GARCH model.

Equally weighted volatility estimator is computed according to next formula:

$$\sigma_t = \sqrt{\frac{1}{T} \sum_{t=1}^T (r_t - \bar{r})^2},$$

where T - sample period, \bar{r} - sample mean. Thus, if in the past extreme event occurred, then the influence of this event will continue with the same weight, and volatility will be high, even if it has returned to normal level long ago. Moreover, after T days of occurrence of this extreme event, the volatility estimate will abruptly jump down as it jumped up before, but there is no apparent reason for this decrease, it is a ghost of event happened T days ago. This feature of equally weighted estimate is known as ghosting feature. Estimates with larger sample periods are more stable than those with smaller periods, as the weight of each observation is smaller, but longer periods may miss variation in volatility. In addition, Alexander and Leigh (1997) point out, that BIS recommend to use the square root of time rule (the variance of t -days returns equals t multiplied by the variance of daily returns) for obtaining forecasts over entire holding period, but the rule is based on assumption of constant volatility that is not observed in reality.

Exponentially weighted moving average (EWMA) enables to solve the problem of ghosting feature. This method is used in RiskMetrics methodology and based on different weights for observations: past returns are given smaller weights according to their position in the data set, recent returns receive higher weights. This framework leads to faster reaction of volatility to shocks, than in case of equally weighted

estimate, and there is no abrupt change in volatility estimate when extreme observation falls out of the sample, as the weight of this observation declines exponentially. Above-described idea is reflected in the formula of volatility estimator:

$$\sigma_t = \sqrt{(1 - \lambda) \sum_{t=1}^T \lambda^{t-1} (r_t - \bar{r})^2},$$

where λ - decay factor lying in the interval $(0, 1)$, T -given sample period. Consequently, the observation t days ago is multiplied by λ^t ; as λ is less than one, observations in the deep past receive very small weights.

Under the assumption of zero sample mean, the recursive formula for EWMA is:

$$\sigma_{t+1|t}^2 = \lambda \sigma_{t|t-1}^2 + (1 - \lambda) r_t^2$$

Thereby, the forecast of volatility for period $t + 1$, given data at time t , is weighted sum of volatility forecast at time $t - 1$ and squared return. In case of multiple assets the covariance has to be calculated, according to this approach it is defined as: $\sigma_{ij}^2 = (1 - \lambda) \sum_{t=1}^T \lambda^{t-1} (r_{it} - \bar{r}_{it})(r_{jt} - \bar{r}_{jt})$. Covariance can also be written in the recursive form.

Monthly (25 trading days) forecasts of volatility and covariance are also derived, but not by smoothing monthly returns, but by smoothing 25-day moving variance estimate, so that monthly variance forecast is written in the following way:

$$\sigma_t^2 = \lambda \sigma_{t-1}^2 + (1 - \lambda) s_t^2,$$

where s_t^2 - 25-day equally weighted variance. Alexander (1997) notices, that this monthly forecast achieve maximum level only 25 days after extreme market event, when moving variance estimate falls down, because extreme observation is not more in the sample period ⁵. Thus, this monthly forecast does not solve the problem of ghosting feature.

Optimal decay factor λ is obtained from the minimization of root mean squared error of variance forecast; in RiskMetrics optimal decay factor for daily data set equals 0.94, for monthly-0.97 (RiskMetrics methodology applies one decay factor for the whole covariance matrix, as for large covariance matrices it is difficult to find decay factors, which correspond to the properties of covariance matrix).

And the third widely used approach is **GARCH model**. Conditional variance of returns follows next process ⁶:

$$\sigma_t^2 = \omega + \beta \sigma_{t-1}^2 + \alpha (r_t - \mu)^2,$$

⁵According to above expression $\sigma_t^2 > \sigma_{t-1}^2 \Leftrightarrow s_t^2 > \sigma_{t-1}^2$

⁶The formula for GARCH (1,1) is presented here, it is most widely used in practice, as many empirical studies showed that inclusion of one lag for volatility and innovation is enough.

where μ -mean portfolio return, ω, β, α are positive constants (this ensures that variance is positive). Parameters of the model can be estimated using maximum likelihood method. GARCH model enables to capture clustering effect: high volatility in the previous period results in high volatility in the next one, as well as low volatility in previous period leads to low volatility in next period. If $\alpha + \beta < 1$, then conditional variance owns the property of mean reversion: after shock it eventually returns to the unconditional mean. It should be noted here, that EWMA in RiskMetrics is special case of GARCH (1,1) model with $\mu = 0, \omega = 0, \alpha = 1 - \beta$ and it does not owe the property of mean reversion ($\alpha + \beta = 1$). It is usually emphasized, that infinite EWMA model is equivalent to Integrated GARCH model without constant ω .

In case of multiple assets it becomes difficult to estimate GARCH model, as number of parameters, that have to be estimated, increases exponentially with the number of series. For example, if portfolio contains two assets, then nine parameters have to be estimated.

Duffie and Pan (1997) point out potential drawback of GARCH model: high current return (as squared value of it is included in the model) can cause instability in parameter estimation, leading to overshooting in the forecast of volatility.

Another possibility to estimate volatility is to use implied volatilities, obtained from equalizing the market price of the option to the model price. This approach accounts for new information in the market, but the menu of traded options is not large enough to provide all necessary data for VaR calculation (in particular, correlations).

4.2 Historical simulation methods

Historical simulation method is one of the easiest to implement and is based on the history of past changes in risk factors over certain period of time (this period of time is also called window), assuming that current portfolio was held also in the past. It means that the hypothetical changes in the portfolio value are constructed on the basis of real past changes in risk factors (as empirical calculations will be made for stocks, then below the method is described in terms of prices).

The scheme of implementing the method is following. First, the hypothetical future prices have to be found. Wiener (1997) points out two ways of applying past changes in data to current prices in order to receive hypothetical future prices: multiplicative and additive approaches. In the first case, the current price is multiplied by the ratio of two adjacent prices at each moment in time in the window, in the second case the difference between two adjacent prices is added to the current price. Multiplicative approach can be used in cases when volatility increases with the increase of price (for example, in case of stock indices, exchange rates), additive approach is suitable for the case of independent volatility level. Thus, after the time-series of hypothetical future prices is received, the possible portfolio values can be found and, consequently, possible relative changes in portfolio value. Then, these changes have

to be ordered from smallest to greatest. In the last step, the change of portfolio value, corresponding to certain level of significance, is defined, and it is VaR. If the percentile appeared to be between two changes of portfolio value, then VaR can be found through interpolation between two adjacent changes.

There are several drawbacks, as well as merits of presented method. One of the most important shortcomings of the method is that it assigns equal weights to all observations in the window (empirical cdf of hypothetical portfolio returns is built in assumption, that probability of each return is reciprocal to number of days in observed period). It means that returns are assumed to be independent and identically distributed over time. But it is not the case in reality, as clustering effects are observed and periods of high and low volatility alternate. If equal weights are used, then the idea, that more distant observations in the past are less informative than recent returns for determining the present risk of portfolio, is lost. Boudoukh, Richardson and Whitelaw (1998) introduce **hybrid approach** of VaR estimation in order to overcome this problem.

The approach is called hybrid as it combines the features of historical simulation method and RiskMetrics model. Recall, that EWMA is used in RiskMetrics methodology, allowing more recent observations to have higher weights in comparison with earlier observations. The same idea is applied by authors to returns in the historical simulation method: more recent returns receive higher weights than more distant, so that different probability weights are now used in order to construct the empirical distribution function. The scheme of method's implementation is following. The time-series of portfolio returns is calculated, then, each return is assigned a weight according to its position in the window. The weights decline exponentially, while moving in the past, and are summed up to 1: for example, if the most recent return gets the probability weight $w(1)$, then next return gets the weight $\lambda \cdot w(1)$, where λ is decay factor ($0 < \lambda < 1$), the third return is assigned the weight $\lambda^2 \cdot w(1)$, and so on. The returns are ordered then from the lowest to the greatest, and VaR is found as certain percentile of the distribution (usually, for given confidence level VaR does not correspond to the observed return of the portfolio, but lies between two returns; then linear interpolation between two adjacent points is used in order to achieve the desired level of significance).

Another model, which captures the clustering effect of returns time-series, is model presented by Barone-Adesi, Giannopoulos and Vosper (1999) and examined by Pritsker (2001). The method is called filtered historical simulation. This approach enables to account for conditional heteroskedasticity of returns without assumption of normal distribution. Returns are assumed to follow GARCH (1,1) process, but the innovations are not drawn from the standard normal distribution. It is assumed, that innovations are i.i.d. with zero mean and unit variance, and the distribution of innovations allow to estimate parameters of GARCH (1,1) process consistently, so there is no assumption about normality of distribution. The hypothetical returns are generated then on the basis of random draws from empirical distribution of innovations, which is obtained from estimation of GARCH model.

Another disadvantage of historical simulation method is that the trade-off between short and long time periods, used for VaR calculation, exists. Longer periods enable to obtain more stable estimates of VaR, on the contrary, short periods (small windows) lead to abrupt shifts in the value of VaR. This result was obtained by Hendricks (1996). Author also points out that it is difficult to obtain accurate estimates of extreme percentiles with small samples. Moreover, if the level of significance is lower than the reciprocal value of number of days in the window, then it is impossible to get estimate of VaR with historical simulation method. In spite of the possibility of longer periods to provide more stable estimates of VaR, the data in the deep past may not be more relevant in present.

Goorbergh and Vlaar (1999) suggest the method of solving the problem of potential impossibility to calculate VaR and the problem of discrete empirical distribution function of portfolio returns, used in historical simulation method (authors point out that, because of using discrete empirical distribution function instead of true one, the bias in results can appear). Authors propose to consider the tail of distribution of returns and to approximate it with Pareto distribution after some threshold level: the tail index can be estimated on the basis of the threshold level, observations greater than threshold level and number of these observations, then for the given probability the quantile of distribution can be estimated.

One of the most important advantages of historical simulation method is that it does not assume certain distribution of returns, so that existed fat tails and other characteristics of the data distribution can be accounted for.

Butler and Schacter (1997) introduce new model of VaR estimation, which is based on the historical simulation method and kernel technique. Authors suggest to estimate the distribution of portfolio returns using kernels (five different kernels were used in the model), then, on the basis of estimated probability density function and cumulative distribution function, they suggest to estimate the distribution of order statistic. Statistic of order j is defined as such value that j data points are below or at this value, and $n - j$ data points are above this value (n -total number of data points). Authors derive the expression for the probability density function of $j - th$ order statistic, it depends on estimated earlier with help of kernel technique pdf and cdf of portfolio returns. The mean and variance of the $j - th$ order statistic can be found by numerical integration (as there is no analytical expression for kernel density estimator). And the mean of the $j - th$ order statistic represents the estimate of VaR.

4.3 Monte Carlo method

Monte Carlo method is a non-parametric method (as historical simulation method) and does not assume the law of distribution of risk factors. It is based on simulations of price paths of variables according to the chosen stochastic models certain number of times (for example, 10 000 times). The commonly used stochastic model for

simulation of assets prices is geometric Brownian motion:

$$dS_t = \mu_t S_t dt + \sigma_t S_t dz,$$

where μ_t, σ_t - instantaneous drift and volatility at time t , $dz \sim N(0, dt)$. The idea of VaR estimation with this method is following: the sequences of assets prices are simulated up to the target horizon and the last prices are used for portfolio estimation, thus the value of portfolio according to one simulation is obtained. Then this procedure is repeated required number of times in order to receive the distribution of portfolio values; in each case the difference between the simulated and initial portfolio value is found. Calculated differences are ordered in ascending order and VaR is found as the percentile corresponding to the desired confidence level.

The accuracy of the method increases with increase in number of replications, but with increase in number of replications increases also the time necessary for calculations. For example, in order to increase the accuracy of calculations 10 times, the number of conducted simulations has to be increased 100 times, because the standard error is reciprocal to the square root of number of replications. Thus, there is a trade-off between speed and accuracy of calculations. One of the most important advantages of the method is that it enables to capture for fat tails, extreme scenarios, it can be used when non-linear instruments are included in portfolio. However, in addition to the feature that the method is time consuming, there is a risk to choose the wrong stochastic model of risk factors behavior ⁷.

In order to perform Monte Carlo method for estimating liquidity-adjusted VaR the model for simulation bid and ask prices has to be chosen. One way of bid prices simulation was presented by Duffie and Ziegler (2001).

Authors investigate the influence of spread on different risk measures, among which there is VaR. Authors consider portfolio, consisting of cash, liquid asset and illiquid asset. The firm owes this portfolio and also some given volume of liabilities. Every period (the period of 10 days is analyzed) certain number of units of each asset is liquidated, obtained money are used to finance the liabilities. Consequently, at the end of the period the firm has new level of liabilities and the portfolio has certain value. The difference between portfolio value and liabilities presents the capital of the company (the ratio of the capital to total assets value must satisfy capital requirements). The initial capital is calculated as difference between portfolio value, which is found using mid-prices, and initial value of liabilities. The capital at the end of the period is calculated in analogous way. The movement of mid-prices of two assets is described with model of geometric Brownian motion. The mid-prices of liquid and illiquid assets at time t are written then in the next way:

$$S_{1,t} = S_{1,0} \exp(\mu_1 t + \sigma_1 B_{1,t}),$$

$$S_{2,t} = S_{2,0} \exp(\mu_2 t + \sigma_2(\rho B_{1,t} + \sqrt{1 - \rho^2} B_{2,t})),$$

⁷With help of sensitivity analysis it can be checked how the results of calculations change with the changes in the model.

where $S_{1,t}, S_{2,t}$ - mid-prices of the liquid and illiquid asset respectively, B_1, B_2 - independent standard Brownian motions, μ_i - instantaneous expected return of the mid-price, σ_i - instantaneous volatility, ρ - instantaneous correlation between returns of two assets. The sale of assets is conducted at the bid price, in order to obtain the process for the bid price the relative mid-to-bid spread is assumed to follow next process (indices 1 and 2 correspond to the liquid and illiquid asset respectively):

$$X_{1,t} = X_{1,0} \exp \left(\gamma_1 (\rho_1 B_{1,t} + \sqrt{1 - \rho_1^2} B_{3,t}) - \frac{1}{2} \gamma_1^2 t \right),$$

$$X_{2,t} = X_{2,0} \exp \left(\gamma_2 (\rho_2 (\rho B_{1,t} + \sqrt{1 - \rho^2} B_{2,t}) + \sqrt{1 - \rho_2^2} B_{4,t}) - \frac{1}{2} \gamma_2^2 t \right),$$

where γ_i - volatility of the relative bid-ask spread for asset i , ρ_i - correlation between change in the spread and asset's return, B_3, B_4 - independent standard Brownian motions (also independent with respect to B_1, B_2), $X_{2,0} > X_{1,0} > 0$ - initial values of spread⁸. Authors apply Monte Carlo method to above-described model and receive the sequence of values of capital at the end of 10 days period, then the difference between these values and initial capital is calculated. Then, authors find the 99%-VaR, expected tail loss and probability of insolvency. Different cases of initial spreads are considered, as well as different values of correlation between spreads and asset returns (VaR and expected tail loss increase with increase in initial levels of spread and with increase in correlation level). In addition, the model is extended to the case of jumps in prices (in order to account fat tails), resulting in higher values of all risk measures. Authors also found out that in case of higher volatility of asset price the relative influence of spread on VaR and expected tail loss declines. It should be mentioned here, that two possible liquidation strategies are considered in the article. According to the first strategy more liquid assets are liquidated first (above-described results relate to this strategy), according to the second - illiquid assets. In the second case VaR and expected tail loss, as it was pointed out before, increase with increase in initial levels of spread and with increase in correlation level, but the values of all three risk measures are lower than in the case of the first strategy. However, the second strategy leads to higher transaction costs.

Thus, three main groups of methods of VaR estimation were described. These methods can be applied to estimation of liquidity-adjusted VaR; in last case (Monte Carlo method) the special model was considered, which can be used for estimation of liquidity-adjusted VaR. But in order to understand what model of estimation is the best one, the backtesting procedure has to be conducted; it is based on comparison of realized losses of portfolio with calculated values of VaR. Next section presents the review of tests, which can be used for estimation of model's accuracy. Some of them will be used later for testing our empirical results.

⁸The bid price can be computed then as $S_{i,t}(1 - X_{i,t})$.

5 Backtesting

According to amendment of 1996 introduced by Basle Committee market risk capital requirement can be calculated by Standard Model approach or Internal Model approach. Standard Model approach is based on building-block methodology, when required capital is calculated separately for each category of market risk, and then obtained volumes are summed up. This method is easy to implement, however, it does not enable to account for effect of diversification, and can be restrictive for banks with effective risk-management. Internal Model approach allows banks to use their own models in order to compute VaR. But not only received number of VaR plays role in determining capital requirements. The performance of the used VaR model is verified over the period of 250 trading days. According to Internal Model approach market risk capital is defined by the following expression:

$$MRC_t = \max(VaR_t, k_t \frac{1}{60} \sum_{i=0}^{59} VaR_{t-i}) + c,$$

where k_t is multiplication factor, which depends on the accuracy of the VaR model, c -capital charge for specific risk. The value of multiplication factor depends on the number of VaR violations in the past: if over last 250 days the number of violations was less or equal than 4, then bank fall in the green zone and $k_t = 3$; if number of violations lies in the interval $[5; 9]$, then k_t increases with increase in number of violations until 3.85 for 9 violations; if number of violations 10 or more than 10, $k_t = 4$. Consequently, the problem of frequency of VaR violations, the accuracy of VaR model is important for the bank, because with increase of multiplication factor k_t the required capital increases. Thus, it will be useful to consider different techniques of evaluation of VaR model, which allow to define whether the used model is accurate one.

In this section five groups of approaches to backtesting the accuracy of VaR model are described: unconditional coverage tests; tests of independence; conditional coverage test; test, based on multiple levels of significance, and loss function model. In each case we will point out advantages and disadvantages of the methods in order to explain the reasons for development of new methods, which account for various aspects that were omitted before.

5.1 Unconditional coverage tests

The first group of the tests is *unconditional coverage tests*. These tests verify whether the failure rate (the ratio of number of observations when losses exceeded VaR to total number of observations) equals the given level of significance, which is used for VaR calculation. If failure rate is higher than level of significance, meaning too frequent VaR violations, then VaR model underestimates risk, if failure rate is lower, then the model is too conservative.

Denote x -number of cases, when VaR is exceeded, T -sample size, p -level of significance under which VaR was computed. Kupiec (1995) suggested the likelihood ratio statistic in order to test null hypothesis $x/T = p$:

$$LR_{uc} = 2[\ln(x/T)^x(1 - x/T)^{T-x} - \ln(p^x(1 - p)^{T-x})]$$

Under the null hypothesis $LR_{uc} \stackrel{\text{as.}}{\sim} \chi^2(1)$. Then, confidence intervals for number of exceptions can be constructed: with how many exceptions for certain confidence level (for example, 95 %), certain sample size T and certain level of significance of VaR calculation the null hypothesis cannot be rejected.⁹

Campbell (2005) points out that the null hypothesis can be verified by Wald test, based on the statistic, which has approximate standard normal distribution (statistic is built using mathematical expectation pT and variance $p(1 - p)T$ of the number of VaR violations).

One of the drawbacks of unconditional coverage models is the low power of the tests for the small sample periods (for example, 250 trading days) and high confidence levels. Low power of the test means insufficient probability of detecting the model of VaR calculation, which underestimates risk, so that the incorrect model is not rejected. The latter leads to the wrong (too low) capital charge for the bank. Campbell (2005) gives a good example of the danger of not detecting incorrect model: in case of normal distribution the capital charge will be about 20 % less than required level, if VaR is reported on the level of significance 3 % instead of required level of 1 % (for level of significance 3 % the critical value of standard normal distribution equals 1.88, for 1 % level of significance-2.33). Thus, low probability of finding out whether the model is incorrect can have negative consequences in the sense of low inadequate market risk capital. The power of the test can be increased by reducing the confidence level or by increasing the number of observations (for example, consideration of 500 days instead of 250 days).

Another drawback of unconditional coverage models is that they implicitly assume independence of excesses of VaR, and, thus, do not enable to verify whether past VaR violations predict future violations or whether they are independent. The possible clustering of VaR violations means that VaR model does not react quickly to the increase in risk. In order to overcome this problem, the conditional coverage models have to be considered.

5.2 Tests of independence

Christoffersen (1998) suggested *likelihood ratio test of independence*. The framework of the model is next one: author considers time series y_t and interval forecasts with lower and upper bounds of the forecast made at time t for the next period for coverage probability p . Then indicator variable I_t is introduced, it indicates whether the

⁹Kupiec (1995) constructed the 95% confidence intervals for x in case of different levels of significance for VaR and different sample sizes.

observation at time t inside the interval forecast, made at time $t - 1$, or not. If y_t is inside the interval forecast, then indicator variable equals one, otherwise equals zero (in case of VaR the open intervals are used, but this does not change the analysis). The indicator sequence is received by comparing values of y_t with the bounds of the forecasts over the sample period. The null hypothesis of serial independence of the indicator sequence is tested against first-order Markov dependence. Author constructs the likelihood function under the null and alternative hypotheses, the estimations of parameters are obtained from the maximization of log-likelihood functions, then, likelihood ratio statistic is built on the basis of received functions and under null hypothesis statistic is asymptotically distributed as $\chi^2(1)$. Under the alternative hypothesis the transition probability matrix has the following form:

$$\mathbf{\Pi}_1 = \begin{pmatrix} 1 - \pi_{01} & \pi_{01} \\ 1 - \pi_{11} & \pi_{11} \end{pmatrix},$$

where

$$\pi_{ij} = P(I_t = j | I_{t-1} = i)$$

Consequently, the likelihood function is written as:

$$L(\mathbf{\Pi}_1; I_1, \dots, I_T) = (1 - \pi_{01})^{T_{00}} \pi_{01}^{T_{01}} (1 - \pi_{11})^{T_{10}} \pi_{11}^{T_{11}},$$

where T_{ij} is number of days when state j occurred after state i in previous day. The ML estimates of transition probabilities are $\hat{\pi}_{01} = \frac{T_{01}}{T_{00} + T_{01}}$ and $\hat{\pi}_{11} = \frac{T_{11}}{T_{10} + T_{11}}$. Under null hypothesis the transition probability matrix has the next form:

$$\mathbf{\Pi}_2 = \begin{pmatrix} 1 - \pi_2 & \pi_2 \\ 1 - \pi_2 & \pi_2 \end{pmatrix},$$

so that value of indicator today is independent of the value of indicator yesterday. Then the likelihood function is written as:

$$L(\mathbf{\Pi}_2; I_1, \dots, I_T) = (1 - \pi_2)^{T_{00} + T_{10}} \pi_2^{T_{01} + T_{11}}$$

and ML estimate of probability is $\hat{\pi}_2 = \frac{T_{01} + T_{11}}{T}$. LR statistic is obtained with respect to the estimated values of likelihood functions:

$$LR_{ind} = 2[\ln(L(\hat{\Pi}_1; I_1, \dots, I_T)) - \ln(L(\hat{\Pi}_2; I_1, \dots, I_T))] \stackrel{\text{as.}}{\approx} \chi^2(1)$$

It should be mentioned here, that presented test enables to detect VaR model for which the indicator sequence is of first-order Markov structure. Christoffersen and Pelletier (2003) suggested duration based test of independence, allowing detection of more general form of dependence. Authors consider time between two VaR violations (this is called no-hit duration, it is measured in days) and build the test, which verifies the null hypothesis that no-hit duration has no memory (the period of time between two VaR violations is independent of how much time passed since last violation) and a mean duration of $1/p$ days, where p is coverage rate.

The accurate VaR model should possess unconditional coverage property and independence property. Above described tests were devoted to detection of inaccurate VaR model only with respect to one of the properties. However, it is useful to look at the test, which enables to define inaccurate VaR model due to failure of one of the properties: joint test of coverage and independence.

5.3 Joint test

Christoffersen (1998) suggested the *joint test of coverage and independence* based on the LR statistic. The null hypothesis of the unconditional coverage test is tested now against the alternative hypothesis in case of LR independence test and LR statistic is written in the following way ¹⁰:

$$LR_{cc} = 2[\ln(L(\hat{\Pi}_1; I_1, \dots, I_T)) - \ln(L(p; I_1, \dots, I_T))] \stackrel{\text{as.}}{\sim} \chi^2(2)$$

Moreover, there is next relationship between LR statistics of unconditional coverage, independence and conditional coverage tests (when the first observation is ignored in test of unconditional coverage):

$$LR_{cc} = LR_{uc} + LR_{ind}$$

5.4 Other tests

Another approach to backtesting the model of VaR calculation is based on the *multiple levels of significance*. The idea is that model has to be accurate at any level of significance, so that VaR violations must correspond to certain levels of significance, in addition the property of independence should hold: VaR violations at all levels of significance have to be independent from each other.

Crnkovic and Drachman (1996) suggest to consider the forecast of probability distribution function (pdf) of the innovation of portfolio value (portfolio value is defined as the sum of deterministic component and innovation, which has certain distribution) and evaluate the quality of forecast. Diebold, Gunther and Tay (1997) proposed the method for defining whether the sequence of density forecasts coincides with the sequence of data generating processes (with true sequence of conditional densities). The method is based on the probability integral transform $z_t = \int_{-\infty}^{y_t} p_t(u) du$, where $p_t(y_t)$ -sequence of density forecasts, y_t -generated series. If sequence of density forecasts coincides with the sequence of data generating processes, then under certain condition $z_t \stackrel{\text{i.i.d.}}{\sim} U(0, 1)$. Thus, the idea is that, if the forecast of pdf of innovation is correct, then VaR measure is accurate. Properties of uniform distribution and independence correspond to previously discussed properties of unconditional coverage and independence. Crnkovic and Drachman propose to test properties of uniform distribution and independence separately, the former property is tested by construction of Kupier statistic.

¹⁰Recall, that $L(p; I_1, \dots, I_T) = p^x(1 - p)^{T-x}$

The scheme of application of described framework in practice is summarized by Jorion (2001) in the following way: the sequence of levels of significance for VaR calculation is selected, then every day VaR is computed for each significance level and next day is compared with realized losses, during all sample period the number of exceptions is calculated, consequently, at the end of the period there is information about number of VaR violations N_i for each level of significance p_i . This number of violations is divided by the total number of observations in each case: $N_i/T = \hat{F}(p_i)$. If the forecast was perfect, then empirical distribution $\hat{F}(p)$ exactly matches p . Kupier statistic is written in the following way:

$$K = \max_i [\hat{F}(p_i) - p_i] + \max_i [p_i - \hat{F}(p_i)]$$

and has known distribution. Crnkovic and Drachman point out, that with number of observations less than 500 test results begin to deteriorate.

And the last approach, that will be briefly described here, is backtesting based on the *loss function*. Lopez (1999 a,b) proposed to use the loss function that in general form can be written in the next way:

$$C_{mt+1} = \begin{cases} f(\epsilon_{t+1}, VaR_{mt}) & \text{if } \epsilon_{t+1} < VaR_{mt} \\ g(\epsilon_{t+1}, VaR_{mt}) & \text{if } \epsilon_{t+1} \geq VaR_{mt} \end{cases}$$

where ϵ_t - portfolio return, VaR_{mt} -VaR corresponding to the certain model m , $f(x, y)$ and $g(x, y)$ are functions such that $f(x, y) \geq g(x, y)$. The idea of last inequality is that losses have to be higher when actual portfolio return is less than VaR. Author gives different examples of loss functions; loss function can be derived in a way, allowing to address the magnitude of exception. The following form is proposed for such loss function:

$$C_{mt+1} = \begin{cases} 1 + (\epsilon_{t+1} - VaR_{mt})^2 & \text{if } \epsilon_{t+1} < VaR_{mt} \\ 0 & \text{if } \epsilon_{t+1} \geq VaR_{mt} \end{cases}$$

Thereby, loss function reflects not only the fact, whether the exception takes place, but also the size of the violation. For the observed sample of T days the sum of losses C_m can be calculated (or the average loss over the period). But on this step the question about the level of obtained value arises: whether received sum of losses is high or low, or whether the average loss corresponds to what would be expected. Consequently, the distribution of C_m has to be considered. The distribution of C_m depends on the distribution of returns of portfolio, so first of all the distribution of returns has to be fitted (normal with some parameters, GARCH process, etc.). Then, the numerical score C_m over the sample period can be found basing on draws (number of draws equals to the sample size of T days) from estimated distribution of returns and corresponding VaR estimates. This procedure is repeated many times, for example, 1 000 times, in order to obtain empirical distribution of numerical score C_m . It should be mentioned here, that lower values of C_m are better. If numerical score is higher than threshold score, then it is a signal to look intently at VaR model. One of the main drawbacks of the model is necessity to assume stochastic model for

returns behavior: higher empirical score (than it would be in case of true stochastic model) leads to appearance of type 1 error. Nevertheless, the approach is useful for comparison of different models of VaR estimation, in addition, the method can alert the regulator when extreme event has occurred (this is not the case for methods, where only the sequence of exceptions is considered).

As in our empirical study of liquidity-adjusted VaR different models are considered, their accuracy has to be tested. To reach this goal Kupiec test, test of independence and joint test were conducted.

6 Empirical analysis

In this section the way and results of empirical analysis are presented. As it was mentioned above, we are interested in the estimation of model of Bangia et al. Different methods of estimation were applied to this model, and it was verified, whether considered models are accurate.

6.1 Data description

In order to study the influence of liquidity component on the worst possible losses for given confidence level, two portfolios were constructed. The first portfolio is consisted of stocks that are included in DAX Index. Thus, this portfolio is highly liquid one and it can be expected that liquidity component will be insignificant for it. Another portfolio is composed from stocks of TecDAX Index, this portfolio is less liquid, consequently, the importance of liquidity component should be higher in this case than in previous one. Three stocks compose the first portfolio: DAI GY Equity (Daimler AG), LHA GY Equity (Deutsche Lufthansa AG) and SIE GY Equity (Siemens AG). The equities that make up the second portfolio are BC8 GY Equity (Bechtle AG), QSC GY Equity (QSC AG) and RPW GY Equity (Repower Systems AG). In order to support the division of above equities on liquid and less liquid it could be useful to look at values of turnover of these stocks: for example, on 06.08.2008 the volume of turnover of DAI GY Equity was 327,25 m euro, of LHA GY Equity-81,02 m euro, of SIE GY Equity-293,85 m euro, while the volume of turnover of BC8 GY Equity was equaled to 507,248 euro; QSC GY Equity-973,931 euro; RPW GY Equity-952,026 euro. These data present only one evidence that second portfolio is less liquid than first one, traded volume in number of stocks can also serve as indicator of market liquidity.

Data present end of day ask, bid and mid prices of equities for the period 09.01.2006-02.01.2008 (data were taken from Bloomberg database). In Appendix graphs for considered equities are presented. Each graph depicts the histogram for every stock and normal distribution with same parameters as the considered series of returns. Thus, the difference between actual distribution and normal distribution is observed. All distributions have fat tails, are peaked and skewed. Descriptive statistics of series of returns are presented in the Table 1 below.

For every equity the hypothesis of normal distribution is rejected (according to Jarque-Bera statistic) on every reasonable level of significance. Consequently, we can expect that results from variance-covariance approach with equally weighted volatility will be inadequate.

It would be also useful to look at distribution of portfolio spread, as it was mentioned above that distribution of spread is characterized by large deviations from normality. The latter causes the necessity of additional estimation of a factor in the model of Bangia et al. In Appendix kernel density estimates of portfolio spreads are presented

statistic	DAI	LHA	SIE	BC8	QSC	RPW
mean	0.00086	0.0007	0.0008	0.001	-0.0002	0.03
median	0.00084	0.001	0.001	0.0004	-0.003	0.0005
maximum	0.07	0.06	0.09	0.08	0.14	0.26
minimum	-0.06	-0.07	-0.06	-0.10	-0.17	-0.15
std.dev.	0.02	0.02	0.02	0.02	0.03	0.03
skewness	0.05	-0.23	0.55	0.39	0.4	1.65
kurtosis	4.06	4.4	6.37	5.45	9.21	16.79
Jarque-Bera	23.54	45.06	262.1	137.97	816.35	4187.7
probability	0.00	0.00	0.00	0.00	0.00	0.00

Table 1: Descriptive statistics of returns series

(Gaussian kernel function was used). On the same graphs normal densities with corresponding parameters are depicted. Distributions have fat tails and are peaked, deviating from normal. ¹¹

6.2 Results

The model of Bangia et al. (1999) was used for estimation of liquidity-adjusted VaR for two portfolios. Different methods, described above, were applied to this model (to both usual VaR and liquidity component):

1. variance-covariance approach with equal weights and EWMA scheme (GARCH model for volatility was not used because of necessity to estimate too many parameters in case of portfolio consisting of three assets)
2. historical method with equal weights and hybrid approach, as it is interesting to study whether hybrid approach gives improvement in our case in comparison with usual historical method, and also how the estimates of liquidity-adjusted VaR with hybrid approach differ from estimates obtained on the basis of other methods
3. Monte Carlo method

The aim of estimation of liquidity-adjusted VaR for two portfolios is to compare the significance of liquidity component (difference between usual VaR and liquidity-adjusted VaR) in two cases; estimation of the model with different methods will

¹¹Portfolio spread is computed on the basis of weighted series of bid and ask prices of shares as relative spread.

enable to choose the best method using backtesting procedure. All results are presented below. The window of 250 returns is used in each case.

6.2.1 Variance-covariance approach

1) Equally weighted volatility

Ordinary VaR for two portfolios was calculated according to the assumption of normal distribution of log-returns: $VaR = \alpha \sqrt{x' \Sigma x}$, where $\alpha = -2.33$, $x = (1/3, 1/3, 1/3)$, Σ -covariance matrix of returns. In order to obtain the worst possible movement in portfolio spread, ask and bid series for portfolio were computed as weighted ask and bid prices of stocks, then relative spread for each day in the sample period was found. On the basis of received series of relative spreads mean relative spread \bar{S} and standard deviation of relative spread $\tilde{\sigma}$ were calculated (window equals 251 days as we consider relative spread). But in order to estimate the liquidity component, the a factor has to be known. The idea of a factor estimation was described in the literature review section. It concerned the single asset case. In our case, portfolio relative spread also presents the univariate series¹², so that the same technique can be applied.

First, the series of worst possible portfolio spreads from historical method with equal weights was found (it will be described later in more details). Then, a parameter was estimated from regression of received worst possible portfolio spreads on the expression $\bar{S} + a\tilde{\sigma}$. As the coefficient before \bar{S} is always one, then the difference between two series: worst possible portfolio spreads and mean relative spreads \bar{S} was considered and regressed on the $a\tilde{\sigma}$ ¹³.

Then liquidity-adjusted VaR was found as difference between previously calculated value of ordinary VaR and one-half of the worst possible spread, obtained on the basis of a estimate. Results for two portfolios are presented on the graphs (see Figures 2-3).

As it was expected, the liquidity component is more significant in case of less liquid portfolio: in case of liquid portfolio VaR and liquidity-adjusted VaR are very close to each other, while in case of less liquid portfolio number of violations reduces if liquidity-adjusted VaR is considered. It should be noted here, that these graphs are useful also for backtesting procedure, because on the y-axis not just realized returns are depicted, but realized returns minus the half of realized relative spread (this is done in order to account that realized losses will be associated not just with movement in mid prices).

¹²We followed the proposition of Bangia et al. in computation of portfolio relative spread.

¹³The regression was estimated without intercept.

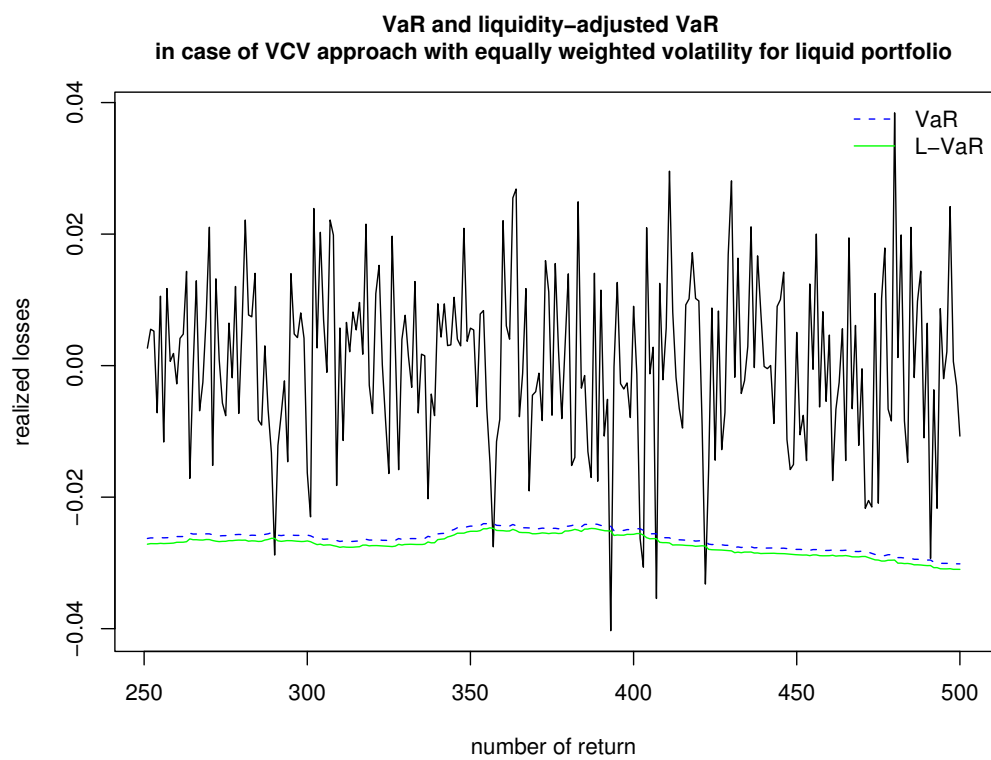


Figure 2: VaR and L-VaR for liquid portfolio, vcv method with equally weighted volatility

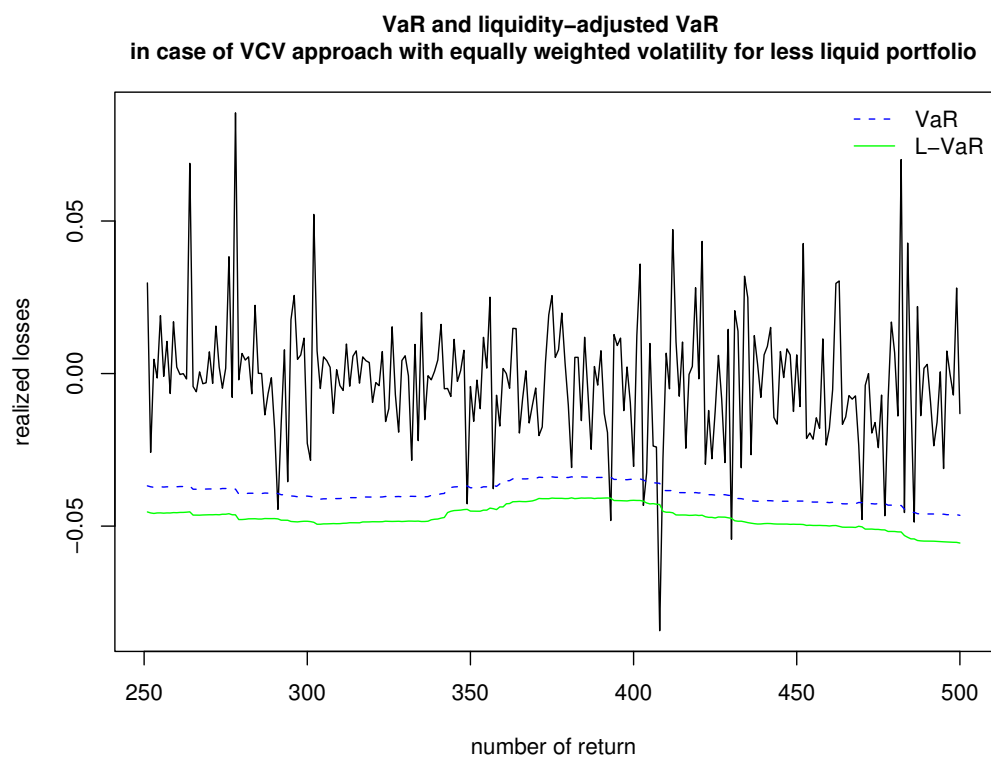


Figure 3: VaR and L-VaR for less liquid portfolio, vcv method with equally weighted volatility

2) EWMA scheme

As equally weighted volatility has the property of ghosting feature, EWMA scheme can be used as improved method. In order to calculate ordinary VaR the whole covariance matrix was estimated on the basis of EWMA scheme (all formulas are presented in the theoretical part), decay factor was taken to be equal to 0.94, according to RiskMetrics methodology. In the next step the worst possible spread has to be found.

The way of its computing is as in the previous case, except that here volatility $\tilde{\sigma}$ is received according to EWMA model. Decay factor $\lambda = 0.96$ was chosen arbitrary. Parameter a was estimated from regression, as in previous case, but for its estimation worst possible spread was used, that was received not from usual historical method with equal weights of observations, but from hybrid method (it will be described later, and also it is described in the theoretical part); also, in this case volatility in regression differs from the previous situation-it is calculated using EWMA.

As the difference in estimates of a factor exists for the cases of equally and EWMA schemes,¹⁴ it could be interesting to compare liquidity-adjusted VaR for two cases: when liquidity component is computed according to equally weighted scheme, and when liquidity component is received from EWMA model. Results are presented on the graphs below (see Figures 4-5): two lower lines (L-VaR1 and L-VaR2) correspond to first and second cases, respectively, blue dashed line presents usual VaR. Two lower lines almost coincide for liquid and less liquid portfolios, some difference appears only at the end for less liquid portfolio. For liquid portfolio it can be concluded with certainty that it does not matter what model to use for defining worst possible spread here. In fact, this can be referred also to less liquid portfolio.¹⁵ But if for liquid portfolio ordinary VaR is very close to liquidity-adjusted VaR, in case of less liquid portfolio this difference is significant and results of backtesting will show the importance of this difference.

¹⁴For example, for less liquid portfolio estimates are 2.9 and 2.5 for the cases of equally weighted and EWMA schemes respectively.

¹⁵In further analysis the second method of liquidity component estimation was used.

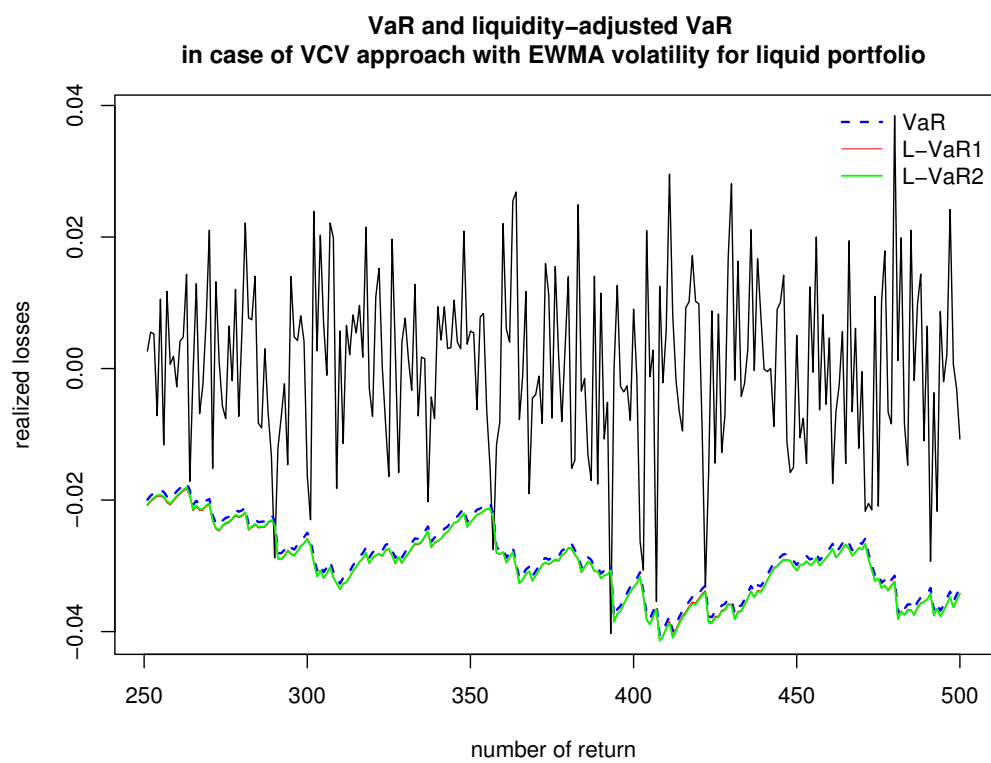


Figure 4: VaR and L-VaR for liquid portfolio, vcv method with EWMA scheme

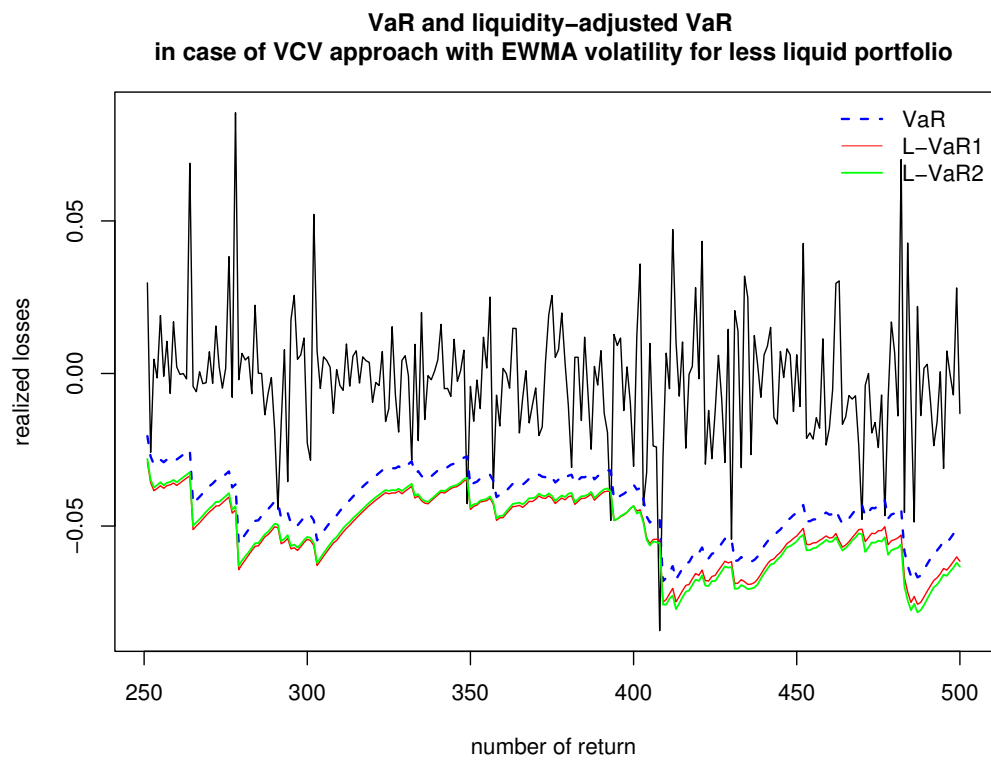


Figure 5: VaR and L-VaR for less liquid portfolio, vcv method with EWMA scheme

6.2.2 Historical simulation approach

1) Equal weights

Another way to obtain liquidity-adjusted VaR is to apply historical simulation method to the ordinary VaR component and to liquidity component. As each return (now returns were calculated without log transformation) receives weight $1/250$, then, due to 1 % level of significance, the third lowest return is ordinary VaR. The same idea of weights distribution was applied to the case of liquidity component: number of observations of portfolio spreads is 251, so each observation receives the weight of $1/251$, then, using interpolation technique, described by Boudoukh, Richardson and Whitelaw (1998),¹⁶ liquidity-adjusted VaR was computed by subtracting the half of worst possible spread from the third lowest return of portfolio. Results for liquid and less liquid portfolio are presented on the graphs (see Figures 6-7). Liquidity component is larger for less liquid portfolio, and also, it can be seen from the graphs, that number of violations is higher for two portfolios than number of violations under EWMA scheme. The latter will be reflected by results of tests in the section devoted to backtesting.

¹⁶Idea of interpolation technique is next one: the weight ($1/251$) is assumed to be evenly distributed between intervals from the observation to the points, defined as middle points between our observation and next lowest observation, and our observation and next highest observation. Basing on this idea and using properties of similarity of triangles, I calculated the worst possible spread for portfolio.

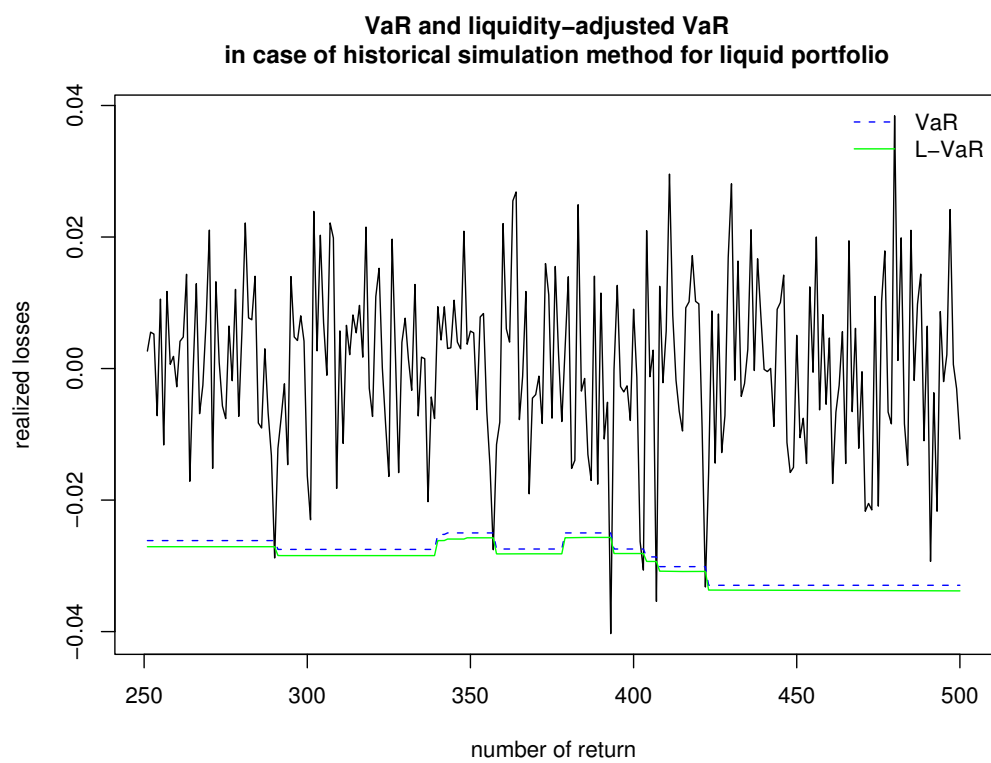


Figure 6: VaR and L-VaR for liquid portfolio, historical method with equal weights

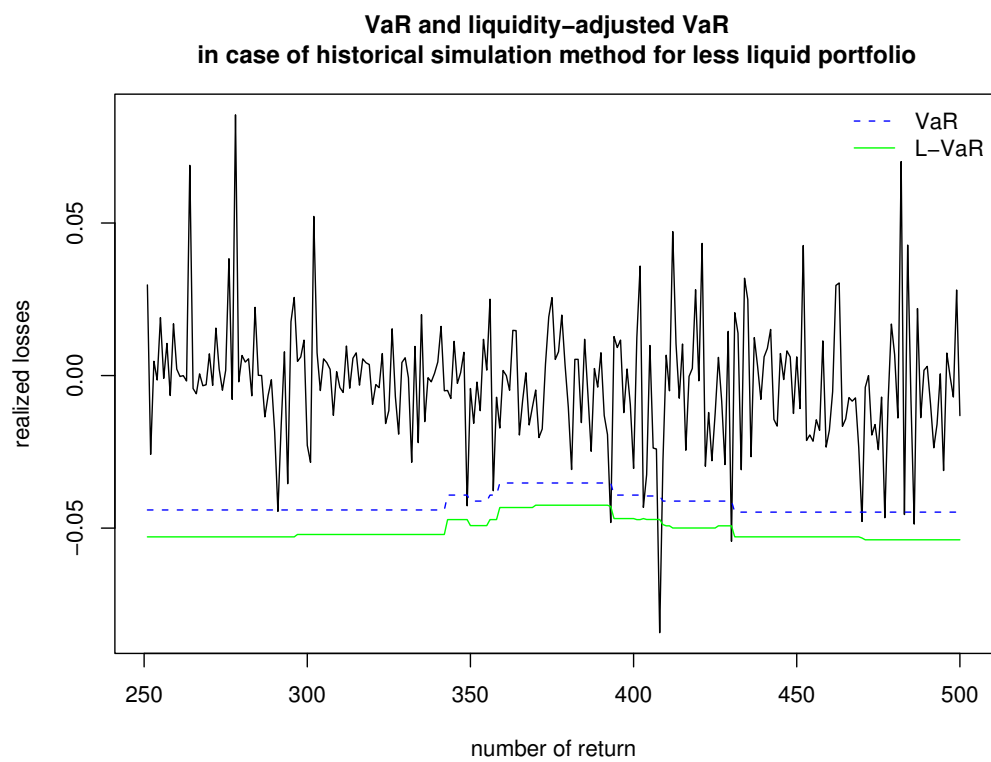


Figure 7: VaR and L-VaR for less liquid portfolio, historical method with equal weights

2) Hybrid method

Hybrid method combines the nonparametric property and feature of putting higher weights to more recent observations. This method was also applied to the model of Bangia et al. Sequence of weights is written in the next way: $(1 - \lambda)/(1 - \lambda^K)$, $((1 - \lambda)/(1 - \lambda^K)) \cdot \lambda$, ..., $((1 - \lambda)/(1 - \lambda^K)) \cdot \lambda^{K-1}$, where K is number of observations (250 in case of returns, 251 in case of spreads). Returns have to be ordered in ascending order (portfolio spreads were also ordered in this way) and the weights have to be accumulated, until the level of significance is not achieved (in case of returns the weights are accumulated, until the level of 0.01 is not reached, in case of spread-until the level of 0.99 is not reached, because we are interested in the big spreads). Then, the above-described idea of interpolation is applied in order to find the worst possible return (worst possible spread). Decay factors for returns and spreads were taken to be equal to 0.94 and 0.96 respectively (the value of decay factor for spread was chosen arbitrary). Results for both portfolios are presented below (see Figures 8-9).

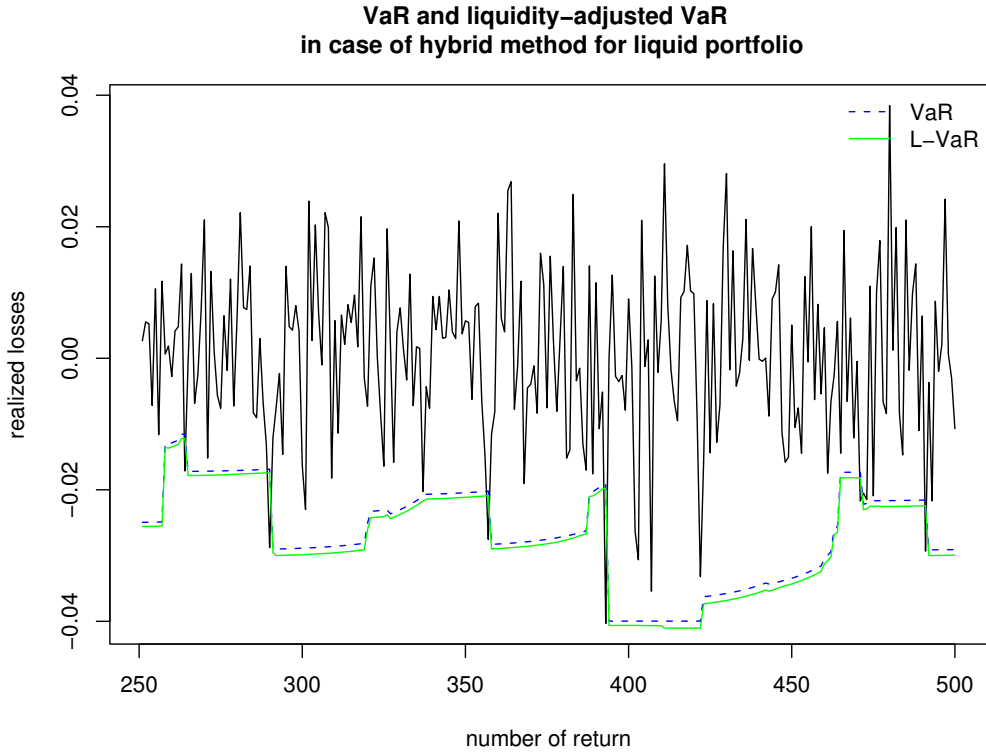


Figure 8: VaR and L-VaR for liquid portfolio, hybrid method

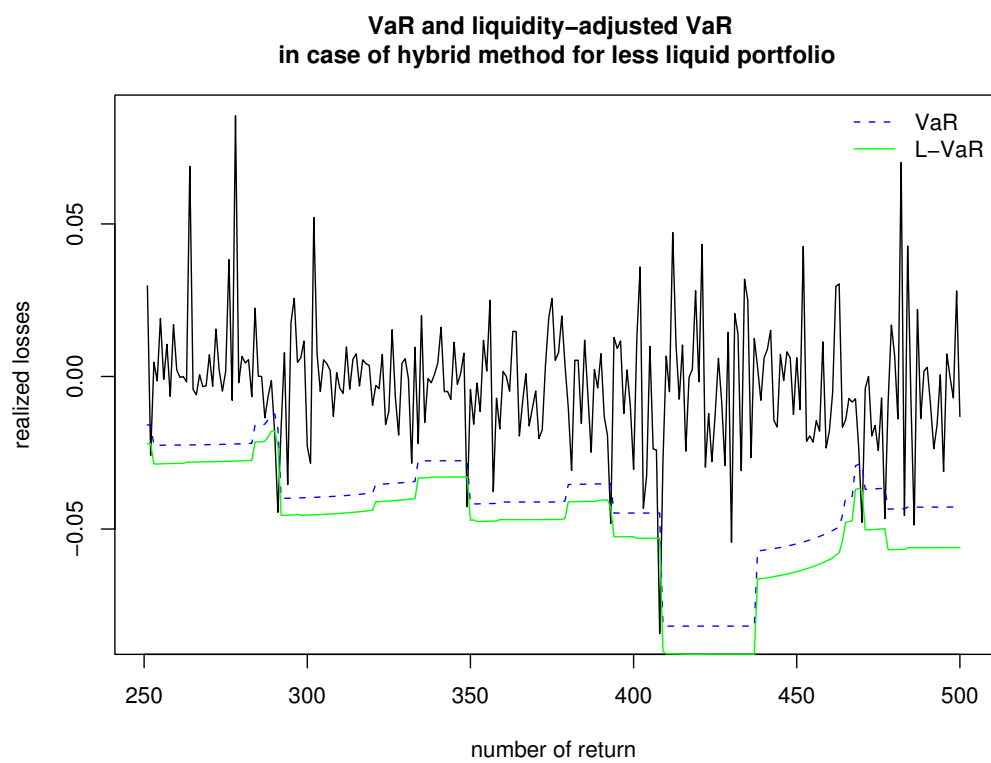


Figure 9: VaR and L-VaR for less liquid portfolio, hybrid method

In comparison with usual historical simulation method the estimations are less stable, but the significance of liquidity component did not change, as it, doubtless, was expected. However, in the sense of number of violations worse results can be expected for this method in comparison with variance-covariance approach with EWMA scheme.

6.2.3 Monte Carlo method

In order to estimate liquidity-adjusted VaR with help of Monte Carlo method, ordinary VaR and liquidity component have to be estimated separately.

In order to estimate the usual VaR 10 000 mid-prices were simulated for each stock in two portfolios. Prices were simulated according to the model of geometric Brownian motion with time horizon of 1 day. Then simulated prices were used for obtaining returns with respect to corresponding initial prices; according to obtained returns portfolio returns were found. To estimate VaR, portfolio returns were putted in ascending order and the observation corresponding to 1 % of the whole sample was found.

But usual VaR is based on the mid-prices, on the other hand, worst possible spread is based on the spread behavior, so that spread has to be simulated. I used the stochastic model from the paper of Duffie and Ziegler (2001), but used it for the case of three stocks¹⁷. The model of these authors was described in the theoretical part as well as equations that are suggested for generation of mid-to-bid spreads. On the basis of simulated mid-to-bid spreads series of bid prices for each stock can be found (using mid-prices that were simulated for ordinary VaR on the previous step), and, consequently, series of ask prices. Then series of relative bid-ask spread for portfolio can be found and worst possible spread is estimated with respect to confidence level of 99 %.

Liquidity-adjusted VaR was computed as difference between usual VaR and half of the worst possible spread. Results for both portfolios are presented below on the graphs (see Figures 10-11).

¹⁷So that Cholesky factorization was applied to correlation matrix of dimension 3×3 .

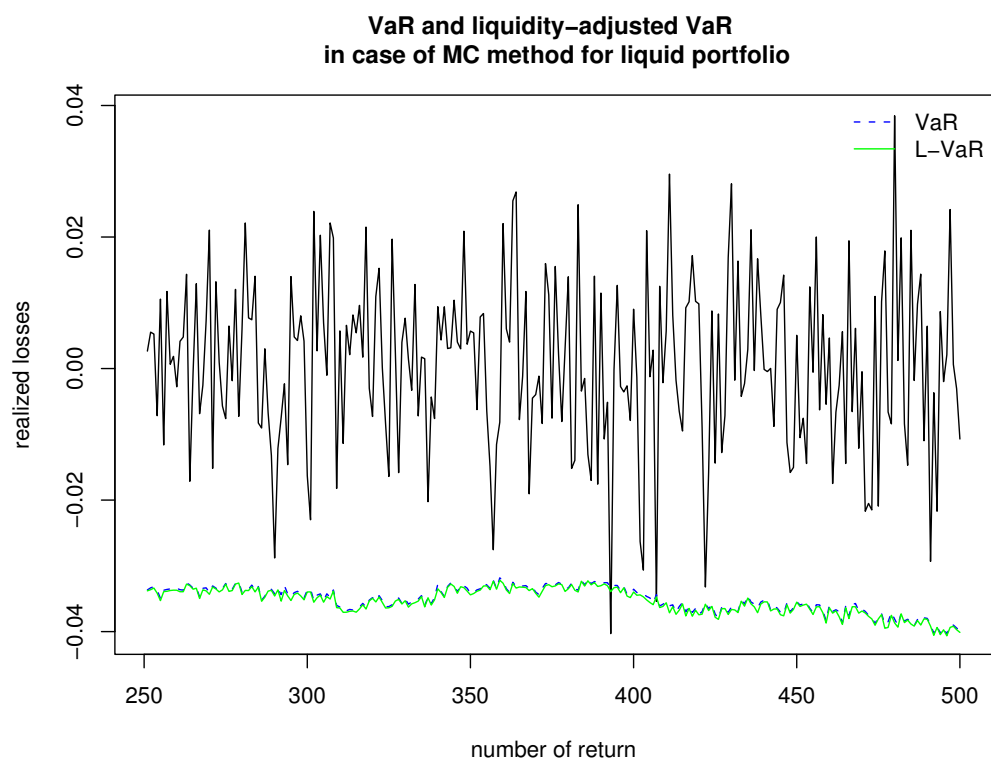


Figure 10: VaR and L-VaR for liquid portfolio, Monte Carlo method

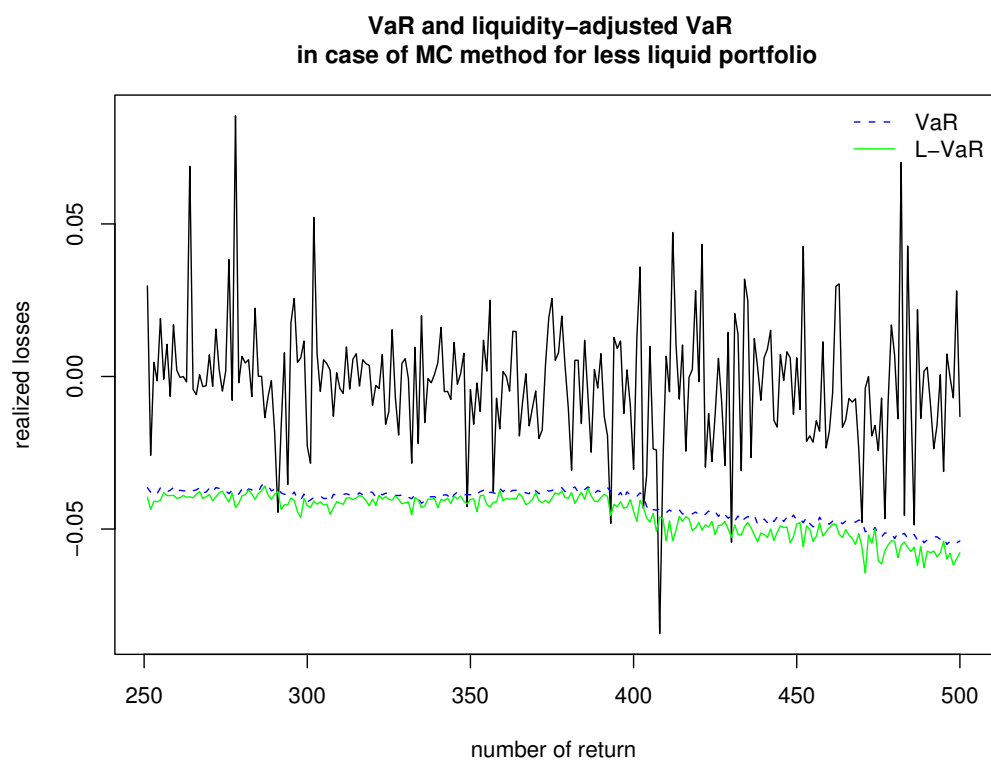


Figure 11: VaR and L-VaR for less liquid portfolio, Monte Carlo method

As we can see, the difference between usual VaR and liquidity-adjusted VaR is not so large for less liquid portfolio as it was in previous cases. Nevertheless, in comparison with liquid portfolio, the difference is higher: for liquid portfolio two lines almost coincide. It means that modified VaR will not give improvement in the number of VaR violations. But the number of violations, as it can be seen from the graph, is low itself, thus there is no necessity of improvement in this case (for liquid portfolio). However, this method of estimation is time consuming, that reduces its attractiveness. In case of less liquid portfolio number of violations reduced with modified VaR, but still, two lines are very close to each other in comparison with other methods. This can be due to specific stochastic model that was chosen for spread simulation. Backtesting procedure will enable to consider all methods in more details and to choose the most adequate model.

6.3 Backtesting results

In theoretical part of work several tests were described, three of them were applied to results of model estimation: unconditional coverage test, test of independence and joint test (it should be noted here, that sum of statistics from two first tests does not equal to the value of statistic from joint test in our case, because test of unconditional coverage was conducted for all observations, including the first one). Critical values on the 95 % confidence level are 3.84, 3.84, 5.99 respectively.

In the tables below (see Tables 2-3) results of tests with values of likelihood ratio statistics are presented, the information concerning the number of violations is presented also on the graph (see Figure 12), showing whether the incorporation of liquidity component in the VaR model allowed to reduce the multiplication factor, which is used for determination of market risk capital requirements.

method			number of vio- lations	Kupiec test LR st.	test of ind-ce LR st.	joint test LR st.
VCV	eq. w.	VaR	7	5.5	1.85	7.4
VCV	eq. w.	Liq-VaR	7	5.5	1.85	7.4
VCV	EWMA	VaR	4	0.77	0.13	0.93
VCV	EWMA	Liq-VaR	3	0.095	0.07	0.19
hist.s.	eq.w.	VaR	6	3.56	0.3	3.9
hist.s.	eq. w.	Liq-VaR	6	3.56	0.3	3.9
hist.s.	hybrid	VaR	6	3.56	0.3	3.9
hist.s.	hybrid	Liq-VaR	6	3.56	0.3	3.9
MC	MC	VaR	2	0.11	0.03	0.16
MC	MC	Liq-VaR	2	0.11	0.03	0.16

Table 2: Results of backtesting for liquid portfolio

method			number of vio- lations	Kupiec test LR st.	test of ind-ce LR st.	joint test LR st.
VCV	eq. w.	VaR	11	15.89	1.02	16.99
VCV	eq. w.	Liq-VaR	4	0.77	0.13	0.93
VCV	EWMA	VaR	9	10.23	0.68	10.98
VCV	EWMA	Liq-VaR	3	0.095	0.07	0.19
hist.s.	eq.w.	VaR	10	12.96	0.84	13.87
hist.s.	eq. w.	Liq-VaR	3	0.09	0.07	0.19
hist.s.	hybrid	VaR	10	12.96	0.71	13.74
hist.s.	hybrid	Liq-VaR	7	5.5	1.85	7.4
MC	MC	VaR	7	5.5	0.41	5.96
MC	MC	Liq-VaR	6	3.56	0.3	3.9

Table 3: Results of backtesting for less liquid portfolio

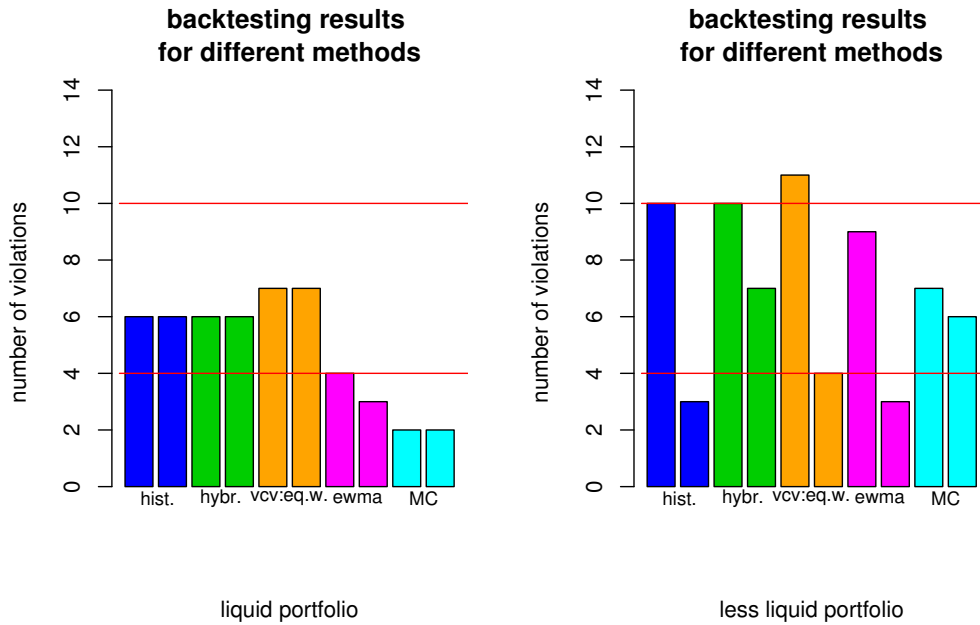


Figure 12: Backtesting results for both portfolios

All methods passed the test of independence: LR statistics for both portfolios in all cases are less than 3.84-critical value on the confidence level of 95%, so that the null hypothesis of the test of independence is not rejected on this confidence level. But some methods turn to be inadequate in the sense of number of VaR violations. Comparison of LR statistics of Kupiec test with 3.84 (critical value on the confidence level of 95%) enables to determine methods, for which the null hypothesis of the Kupiec test is rejected: for example, for liquid portfolio the null hypothesis is rejected only in case of variance-covariance method with equally weighted volatility, but for less liquid portfolio it is rejected more frequently and only in case of liquidity-adjusted VaR, received on the basis of certain methods, it is not rejected. Whether the null hypothesis of the joint test is rejected can be defined in the same way using critical value 5.99.

Variance-covariance approach with equally weighted volatility provides the highest number of violations for both portfolios (if we consider ordinary VaR), leading to the highest value of multiplication factor (red zone) for less liquid portfolio. Inclusion of liquidity component allows less liquid portfolio to move to green zone, but liquid portfolio remains to stay in yellow zone. The latter depicts once more the insignificance of liquidity component for such liquid portfolio. Moreover, only variance-covariance approach with EWMA scheme enables to reduce number of violations due to VaR modification (for *liquid portfolio*). However, even without this improvement variance-covariance approach with EWMA scheme leads to the green zone. Historical and hybrid methods, as well as variance-covariance approach with equally weighted volatility, result in yellow zone, liquidity-adjusted VaR does not enable to move to green zone. It is also interesting that number of violations for historical and hybrid methods are the same. The smallest number of violations is observed in case of Monte Carlo method for both VaR and liquidity-adjusted VaR.

Situation differs for *less liquid portfolio*. Only two of five methods lead us to yellow zone (if we consider ordinary VaR): variance-covariance approach with EWMA scheme and Monte Carlo method, other three methods result in red zone. But inside the yellow zone Monte Carlo method results in lower value of multiplication factor than EWMA method. However, three methods allow to move to green zone, if liquidity component is included: variance-covariance approach with EWMA scheme, with equally weighted volatility and historical method. Hybrid approach and Monte Carlo method reduce the number of violations, but lead to (remain in) yellow zone. Thus, liquidity-adjusted VaR allows to decrease the multiplication factor for less liquid portfolio from highest value to the lowest value in case of two methods. It can be concluded, that variance-covariance approach with EWMA scheme is one of the best methods for two portfolios, nevertheless, in case of liquidity-adjusted VaR other methods also become adequate for our data. Monte Carlo method gives good results for ordinary VaR, while for liquidity-adjusted VaR there is no much improvement.

7 CVaR: concept and estimation

One of the drawbacks of VaR as risk measure is that this measure is not coherent in the sense of coherence defined by Artzner et al. (1999). Artzner et al. (1999) define coherent risk measure as risk measure $\rho(X)$, satisfying the four axioms: axiom of translation invariance, subadditivity, positive homogeneity and monotonicity. Each of these axioms is presented below.

1) Translation invariance: for all random variables $X \in \Theta$ ¹⁸ and all real numbers α ,

$$\rho(X + \alpha r) = \rho(X) - \alpha$$

2) Subadditivity: for all X_1 and $X_2 \in \Theta$, $\rho(X_1 + X_2) \leq \rho(X_1) + \rho(X_2)$

3) Positive homogeneity: for all $\lambda > 0$ and all $X \in \Theta$, $\rho(\lambda X) = \lambda \rho(X)$

4) Monotonicity: for all X and $Y \in \Theta$ with $X \leq Y$, $\rho(Y) \leq \rho(X)$

The property of translation invariance means that if sure amount α is added to the initial position and this amount is invested in the reference instrument, then risk measure is decreased by α . The property of subadditivity supports the idea of portfolio diversification: if this condition is not satisfied, then the risk can be reduced by splitting the portfolio. The measure should be invariant to the money units, in which it is measured (the property of positive homogeneity provides this feature).

VaR does not hold the property of subadditivity, this led to search of other risk measures that satisfy the axioms and are coherent.

The concept of conditional VaR (CVaR) is briefly described below. CVaR is coherent risk measure, but in order to understand better its definition we will follow the paper of Rockafellar and Uryasev (2001), where the general concept of CVaR is developed. Authors distinguish between upper CVaR ($CVaR^+$), lower CVaR ($CVaR^-$) and CVaR, among which only CVaR is coherent risk measure¹⁹. Upper CVaR (it is also called expected shortfall by authors) is defined as expected losses strictly exceeding VaR, lower CVaR (also called tailVaR by authors) present expected losses which are equal to or exceed VaR, while CVaR is determined as weighted average between VaR and upper CVaR.

¹⁸ Θ is the set of all real valued functions on the set of states of nature Ω , X is final net worth of the position for each element of Ω .

¹⁹There is some confusion in the terms in this area: Artzner et al. (1999) call tail conditional expectation (TCE) also tailVaR, Acerbi and Tasche (2002) refer to expected shortfall and CVaR as to the same object and point out, that in general case expected shortfall differ from TCE. But we will follow here the definitions of Rockafellar and Uryasev (2001), according to whom expected shortfall is the upper CVaR and tailVaR is the lower CVaR.

Formally the idea of weighted CVaR is presented in the next way. Loss function $z = f(x, y)$ is assumed to depend on the decision vector $x = (x_1, \dots, x_n)$ and random vector $y = (y_1, \dots, y_m)$; the distribution function for the loss is given by $\Psi(x, \varsigma) = P\{y|f(x, y) \leq \varsigma\}$. Then authors define VaR for the given confidence level as $\varsigma_\alpha(x) = \min\{\varsigma|\Psi(x, \varsigma) \geq \alpha\}$. CVaR is defined as the mean of the tail distribution that is determined by level of confidence α (the definition is taken from the article Rockafellar and Uryasev (2001)):

$\phi_\alpha(x)$ = mean of the α -tail distribution of $z = f(x, y)$, where the distribution in question is the one with distribution function $\Psi_\alpha(x, \cdot)$ defined by:

$$\Psi_\alpha(x, \varsigma) = \begin{cases} 0 & \text{for } \varsigma < \varsigma_\alpha(x) \\ u & \text{for } \varsigma \geq \varsigma_\alpha(x) \end{cases}$$

where $u = [\Psi(x, \varsigma) - \alpha]/[1 - \alpha]$.

If there is a jump in the distribution function at the point $\varsigma_\alpha(x)$, then the problem of defining the α -tail distribution arises, this is the reason for introducing the previous distribution function.

According to above notations, the upper and lower CVaR are defined in the following way:

$$\alpha - CVaR^+ : \phi_\alpha^+(x) = E\{f(x, y)|f(x, y) > \varsigma_\alpha(x)\}$$

$$\alpha - CVaR^- : \phi_\alpha^-(x) = E\{f(x, y)|f(x, y) \geq \varsigma_\alpha(x)\}$$

Moreover, it should be mentioned that upper CVaR is defined only in the case when $P\{f(x, y)|f(x, y) > \varsigma_\alpha(x)\} > 0$.

Authors derive next relations between above-described measures: if there is no jump in the distribution function at point $\varsigma_\alpha(x)$, then $\phi_\alpha^-(x) = \phi_\alpha(x) = \phi_\alpha^+(x)$. So that TCE is coherent risk measure in case of continuous distributions.

If there is a jump and $\Psi(x, \varsigma_\alpha(x)^-) < \alpha < \Psi(x, \varsigma_\alpha(x)) < 1$, then $\phi_\alpha^-(x) < \phi_\alpha(x) < \phi_\alpha^+(x)$, where $\Psi(x, \varsigma_\alpha(x)^-)$ and $\Psi(x, \varsigma_\alpha(x))$ correspond to the lower and upper end-points of the gap, caused by the jump in the distribution function.

On the basis of previous relations authors derive the formula for CVaR as weighted average between VaR and upper CVaR. If $\lambda_\alpha(x)$ is denoted as $\lambda_\alpha(x) = [\Psi(x, \varsigma_\alpha(x)) - \alpha]/[1 - \alpha]$, then

$$\phi_\alpha(x) = \lambda_\alpha(x)\varsigma_\alpha(x) + [1 - \lambda_\alpha(x)]\phi_\alpha^+(x)$$

It should be noted here, that in case $\lambda_\alpha(x) = 1$, reflecting the fact that VaR is highest loss that can occur (upper CVaR is not defined in this case, as it was mentioned before), $\phi_\alpha(x) = \varsigma_\alpha(x)$.

In our case on the basis of different methods values of VaR and liquidity-adjusted VaR were computed. As we are interested in comparison of the value of CVaR for two portfolios, results of only one method are used for computation of CVaR. In

addition, it is interesting to consider, how the magnitude of conditional mathematical expectation will change in both cases, if instead of ordinary VaR liquidity-adjusted VaR is used.

CVaR was estimated as average value of returns that are lower than VaR (VaR from variance-covariance approach with equally weighted volatility was used²⁰). Obtained results confirm the expected results that for less liquid portfolio the value of CVaR should be less (in negative terms) than for liquid portfolio. If instead of usual VaR liquidity-adjusted VaR is used, then value of the average becomes even lower than in case of ordinary VaR, that means that the most severe losses are concentrated in the tail. All results are presented on the graph below: the magnitude of change in the estimation of conditional mathematical expectation is higher for less liquid portfolio than for liquid portfolio (two lower lines correspond to conditional mathematical expectation in case of VaR and liquidity-adjusted VaR for less liquid portfolio respectively, analogous two upper lines are for liquid portfolio).

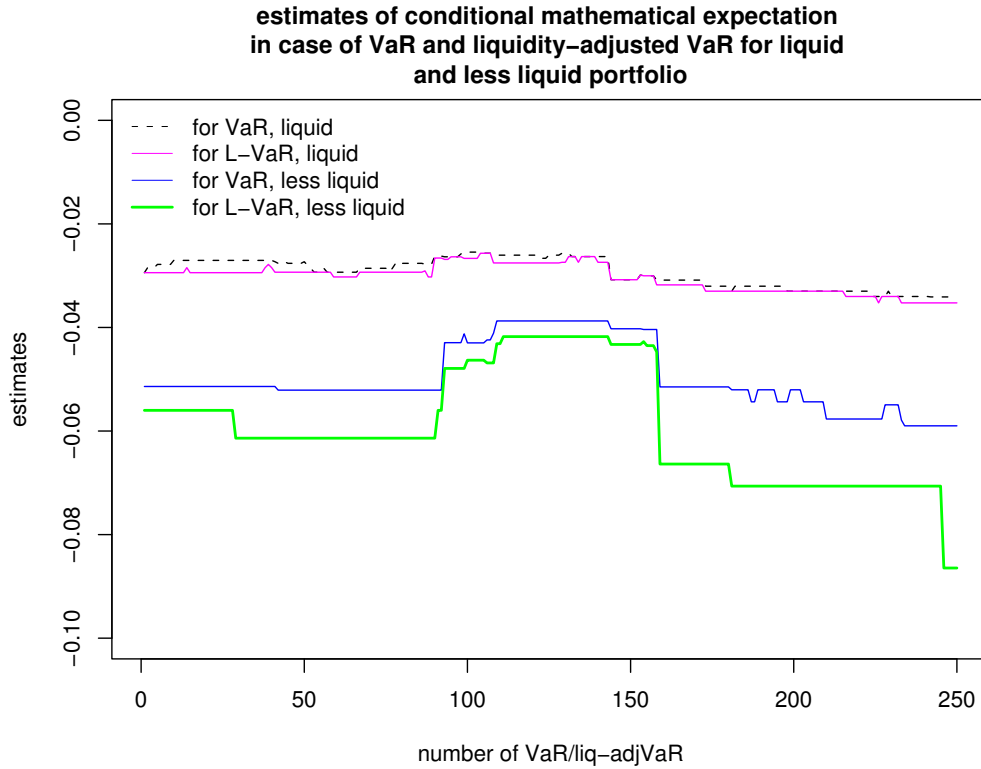


Figure 13: Estimates of conditional mathematical expectation for two portfolios

²⁰It was accounted for mathematical expectation of returns in VaR, so that absolute VaR was considered.

8 Conclusion

In last years the use of VaR measure increased greatly; with introduction of the market risk capital requirements, based on it, methods of VaR estimation became even more important, as the accuracy of model influences market risk capital requirements. But, except methods of estimation, there is also another dimension, that can make the model more adequate: incorporation of liquidity risk into VaR model appeared to result in significant reduction of number of violations. This happens, because mid prices are used for VaR computation, regardless the fact that real price of transaction deviates from the mid price.

Two types of liquidity risk, reflecting different liquidity characteristics, can be introduced into VaR model. In our work we estimated the model, incorporating exogenous liquidity risk into VaR framework.

As there are variety of methods, according to which VaR can be evaluated, three main methods, as well as some of their modifications, were used for model estimation: variance-covariance method with equally weighted and EWMA schemes, historical simulation and hybrid methods, Monte Carlo method. Analysis was applied to highly liquid and less liquid portfolios in order to compare the significance of changes in VaR due to liquidity component.

As it was expected, liquidity component in case of less liquid portfolio is always higher than in case of liquid portfolio. Another aim of using different methods is to compare them in the sense of accuracy for VaR and liquidity-adjusted VaR and to choose the best model of estimation. For that purpose three tests were conducted: unconditional coverage test, test of independence and joint test. All methods passed test of independence, but results differ greatly if ratio of violations is considered. For example, only one method for liquid portfolio enables to reduce number of violations in case of liquidity-adjusted VaR, but for less liquid portfolio all methods decrease number of violations in case of liquidity-adjusted VaR, and some methods even enable to move from red zone to green one. Variance-covariance approach with EWMA scheme turned to be one of the best methods for both portfolios and for both measures. Nevertheless, some methods give great improvement in case of liquidity-adjusted VaR for less liquid portfolio.

In addition, CVaR was estimated for variance-covariance method with equal weights. The value of the measure is higher for less liquid portfolio.

Thus, we can conclude that extension of VaR model in order to introduce liquidity risk in its framework can have significant effect on market risk capital requirements, especially if the portfolio is not highly liquid. It could be interesting to estimate the model, accounting for endogenous liquidity risk, or use other methods of estimation (for example, another stochastic model for spread in Monte Carlo method).

9 Appendix

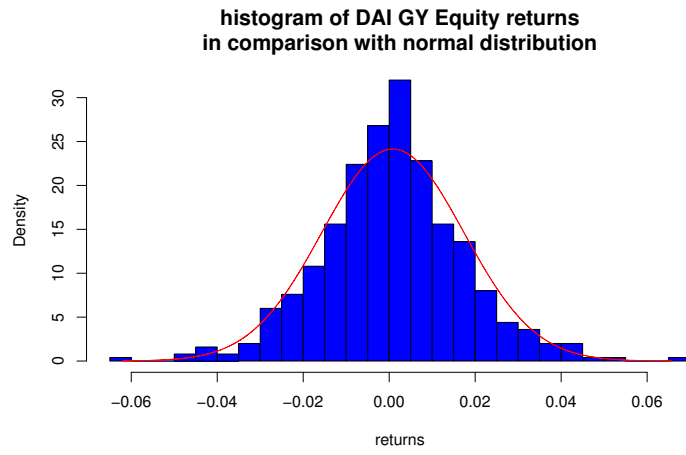


Figure 14: Histogram of DAI GY Equity returns and normal distribution

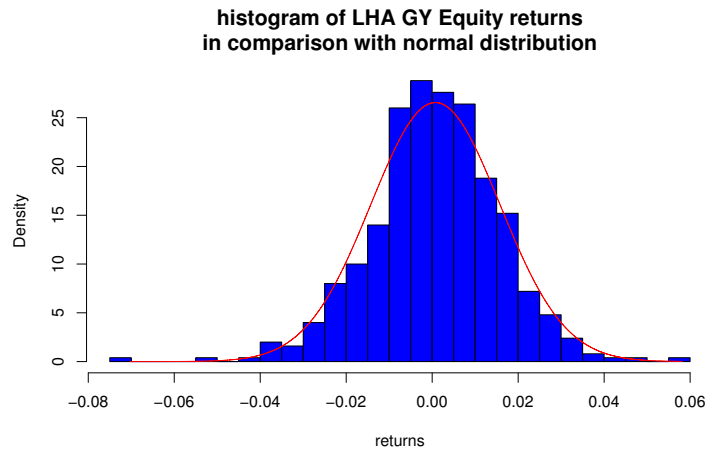


Figure 15: Histogram of LHA GY Equity returns and normal distribution

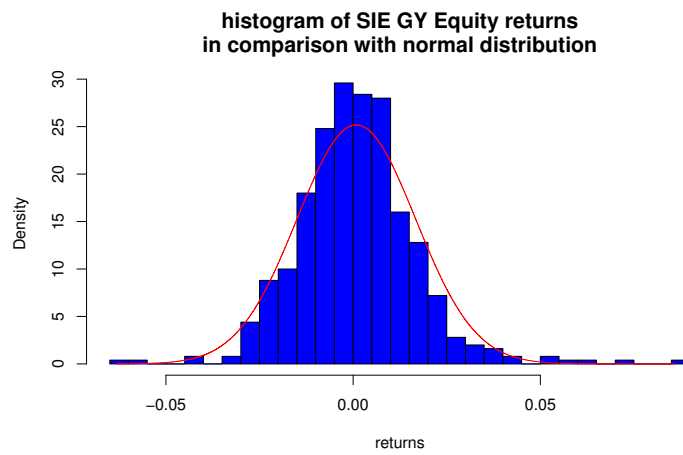


Figure 16: Histogram of SIE GY Equity returns and normal distribution

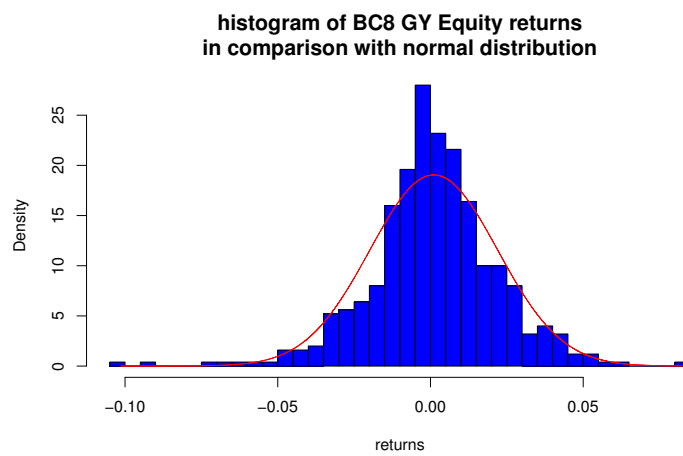


Figure 17: Histogram of BC8 GY Equity returns and normal distribution

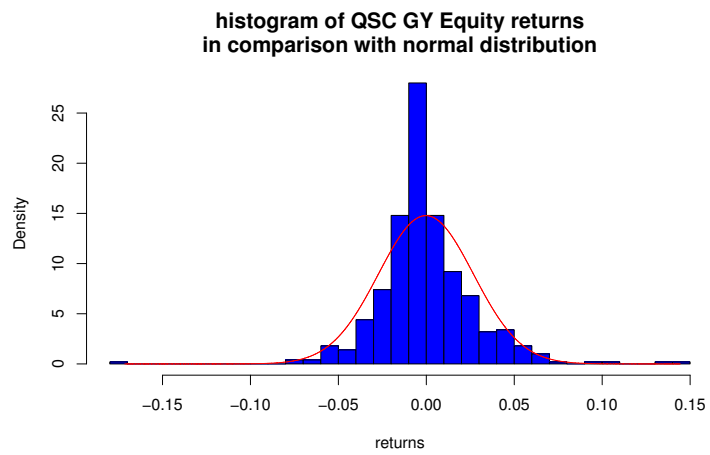


Figure 18: Histogram of QSC GY Equity returns and normal distribution

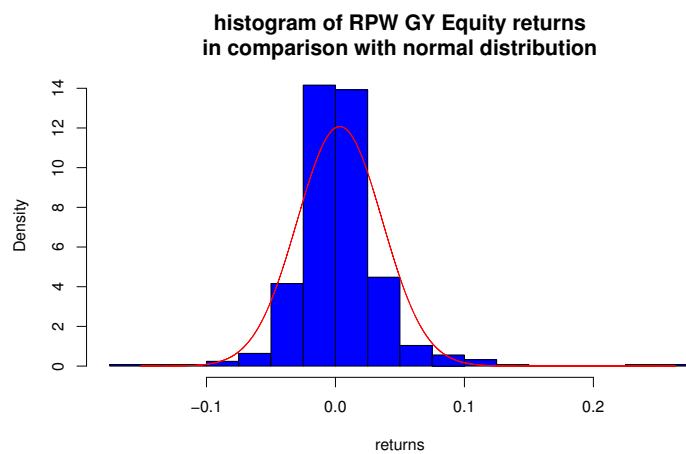


Figure 19: Histogram of RPW GY Equity returns and normal distribution

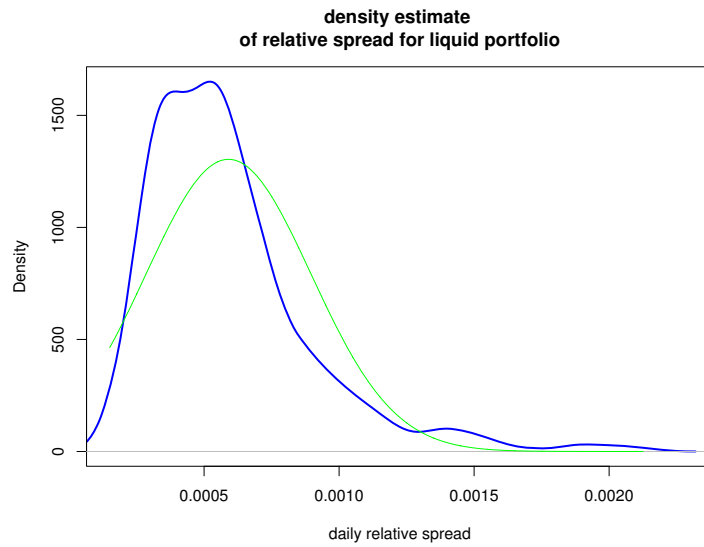


Figure 20: Parametric (normal, thin line) vs nonparametric density estimate of relative spread for liquid portfolio

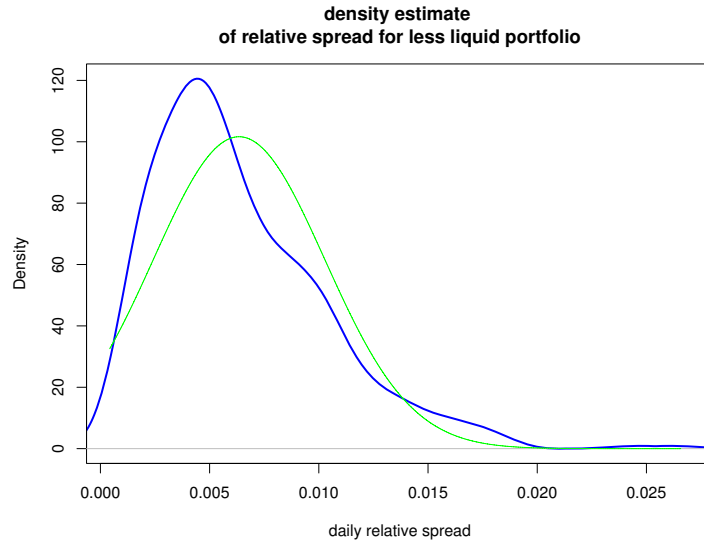


Figure 21: Parametric (normal, thin line) vs nonparametric density estimate of relative spread for less liquid portfolio

10 Bibliography

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Declaration of Authorship

I hereby confirm that I have authored this master thesis independently and without use of others than the indicated sources. All passages which are literally or in general matter taken out of publications or other sources are marked as such.

Berlin, 12 September, 2008,

Ekaterina Orlova